Constraint Satisfaction Problems
CompSci 270

Ron Parr
Department of Computer Science
Duke University

With thanks to Kris Hauser for some slides

Checking for Solution Existence

• In some problems, we don’t care about a path, but about a configuration that has a desired property
• Instead of a goal, we have a target, which can be a set of states that satisfy some property

• We call the set of properties that legal solutions must obey constraints
• We call these problems constraint satisfaction problems (CSPs)
CSP Examples

- Satisfying curriculum/major requirements
- Sudoku
- Seating arrangements at a party
- LSAT Questions: https://www.petersons.com/articles/lsat/sample-lsat-test-questions

CSPs

- Specifying CSPs
- One view: Search with special goal criteria
- CSP definition (general):
  - Variables $X_1, \ldots, X_n$
  - Variable $X_i$ has domain $D_i$
  - Constraints $C_1, \ldots, C_m$
  - Solution: Each variable gets a value from its domain such that no constraints violated
- CSP examples...
  - http://www.csplib.org/
CSP Example

Graph coloring:

Western Australia (WA)
Northern Territory (NT)
Queensland (Q)
South Australia (SA)
New South Whales (NSW)
Tasmania (T)
Victoria (V)

Problem: Assign **Red**, **Green** and **Blue** so that no 2 adjacent regions have the same color. (3-coloring)

CSP as a Search Problem

- **n variables** $X_1, ..., X_n$
- **Valid assignment**: \{$X_{i1} \leftarrow v_{i1}, ..., X_{ik} \leftarrow v_{ik}$\}, $0 \leq k \leq n$, such that the values $v_{i1}, ..., v_{ik}$ satisfy all constraints relating the variables $X_{i1}, ..., X_{ik}$
- **Complete assignment**: one where $k = n$ [if all variable domains have size $d$, there are $O(d^n)$ complete assignments]
- **States**: valid assignments
- **Initial state**: empty assignment {}, i.e. $k = 0$
- **Successor** of a state: 
  \{$X_{i1} \leftarrow v_{i1}, ..., X_{ik} \leftarrow v_{ik}$\} $\rightarrow$ 
  \{$X_{i1} \leftarrow v_{i1}, ..., X_{ik} \leftarrow v_{ik}, X_{ik+1} \leftarrow v_{ik+1}$\}
- **Goal test**: $k = n$
Backtracking Search

- Essentially a simplified depth-first algorithm using recursion

Assignment = {}
Backtracking Search
(3 variables)

Assignment = \{(X_1, v_{11})\}

Backtracking Search
(3 variables)

Assignment = \{(X_1, v_{11}), (X_3, v_{31})\}
Backtracking Search
(3 variables)

Assignment = \{(X_1,v_{11}), (X_3,v_{31})\}

Assume that no value of $X_2$ leads to a valid assignment

Then, the search algorithm backtracks to the previous variable ($X_3$) and tries another value.

Assignment = \{(X_1,v_{11}), (X_3,v_{32})\}
Backtracking Search
(3 variables)

Assumption = \{ (X_1, v_{11}), (X_3, v_{32}) \}

The search algorithm backtracks to the previous variable (X_3) and tries another value. But assume that X_3 has only two possible values. The algorithm backtracks to X_1.

Assume again that no value of X_2 leads to a valid assignment.

Assignment = \{ (X_1, v_{12}) \}
Backtracking Search
(3 variables)

Assignment = \{(X_1,v_{12}), (X_2,v_{21})\}

The algorithm need not consider the variables in the same order in this sub-tree as in the other

Assignment = \{(X_1,v_{12}), (X_2,v_{21})\}
Backtracking Search
(3 variables)

Assignment = \{(X_1, v_{12}), (X_2, v_{21}), (X_3, v_{32})\}

The algorithm need not consider the values of \(X_3\) in the same order in this sub-tree
Backtracking Search
(3 variables)

Assignment = \{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}

Since there are only three variables, the assignment is complete

Backtracking Algorithm

CSP-BACKTRACKING(A)
1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. If A is valid then
      i. result ← CSP-BACKTRACKING(A)
      ii. If result ≠ failure then return result
   c. Remove (X←v) from A
5. Return failure
Efficiency of CSP-Backtracking

CSP-BACKTRACKING(A)
1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X ← v) to A
   b. If a is valid then
      i. result ← CSP-BACKTRACKING(A)
      ii. If result ≠ failure then return result
   c. Remove (X ← v) from A
5. Return failure

Practical Efficiency of CSP Algorithms

• Fundamental trade off
  – Time spent ruling out bad/impossible choices
  – Time spent searching

• Try to find the sweet spot where you quickly rule out bad/unpromising choices
CSP Example Revisited

Graph coloring:

Western Australia (WA)
Northern Territory (NT)
Queensland (Q)
South Australia (SA)
New South Whales (NSW)
Victoria (V)
Tasmania (T)

Problem: Assign Red, Green and Blue so that no 2 adjacent regions have the same color. (3-coloring)

Example Contd.

• Variables: \{WA, NT, Q, SA, NSW, V, T\}
• Domains: \{R,G,B\}
• Constraints:
  For WA – NT:\{(R,G), (R,B), (G,B), (G,R), (B,R), (B,G)\}
• We have a table for each adjacent pair
• Note: Many possible ways to express constraints
Forward Checking in Map Coloring

Forward checking removes the value Red of NT and of SA
Forward Checking in Map Coloring

<table>
<thead>
<tr>
<th></th>
<th>WA</th>
<th>NT</th>
<th>Q</th>
<th>NSW</th>
<th>V</th>
<th>SA</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>GB</td>
<td>G</td>
<td>RGB</td>
<td>RGB</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
</tbody>
</table>

Forward Checking in Map Coloring

<table>
<thead>
<tr>
<th></th>
<th>WA</th>
<th>NT</th>
<th>Q</th>
<th>NSW</th>
<th>V</th>
<th>SA</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>B</td>
<td>G</td>
<td>RB</td>
<td>RGB</td>
<td>B</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>B</td>
<td>G</td>
<td>RB</td>
<td>B</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
</tbody>
</table>
Forward Checking in Map Coloring

Empty set: the current assignment 
\{(WA \leftarrow R), (Q \leftarrow G), (V \leftarrow B)\}
does not lead to a solution

<table>
<thead>
<tr>
<th>WA</th>
<th>NT</th>
<th>Q</th>
<th>NSW</th>
<th>V</th>
<th>SA</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>GB</td>
<td>RGB</td>
<td>RGB</td>
<td>RGB</td>
<td>GB</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>B</td>
<td>G</td>
<td>RB</td>
<td>RGB</td>
<td>B</td>
<td>RGB</td>
</tr>
<tr>
<td>R</td>
<td>B</td>
<td>G</td>
<td>RB</td>
<td>B</td>
<td>RGB</td>
<td></td>
</tr>
</tbody>
</table>

Forward Checking (General Form)

Whenever a pair \((X \leftarrow v)\) is added to assignment \(A\) do:

For each variable \(Y\) not in \(A\) do:

For every constraint \(C\) relating \(Y\) to the variables in \(A\) do:

Remove all values from \(Y\)’s domain that do not satisfy \(C\)
Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)
1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X∈v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X∈v) from A
5. Return failure

No need any more to verify that A is valid
Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)
1. If assignment A is complete then return A
2. X \leftarrow \text{select a variable not in A}
3. D \leftarrow \text{select an ordering on the domain of } X
4. For each value v in D do
   a. Add (X\leftarrow v) to A
   b. var-domains \leftarrow \text{forward checking}(\text{var-domains, X, v, A})
   c. If no variable has an empty domain then
      (i) result \leftarrow \text{CSP-BACKTRACKING}(A, \text{var-domains})
      (ii) If result \neq \text{failure} then return result
   d. Remove (X\leftarrow v) from A
5. Return failure

Need to pass down the updated variable domains
1) Which variable $X_i$ should be assigned a value next?
   → Most-constrained-variable heuristic
   → Most-constraining-variable heuristic

2) In which order should its values be assigned?
   → Least-constraining-value heuristic

NOTE: Different use of the word “heuristic” from A*
Don’t confuse these two! You will only get questions about heuristics as functions from states to reals!

---

**Most-Constrained-Variable Heuristic**

1) Which variable $X_i$ should be assigned a value next?

   Select the variable with the smallest remaining domain
   
   [Rationale: Minimize the branching factor]
Map Coloring

- SA’s remaining domain has size 1 (value B remaining)
- Q’s remaining domain has size 2
- NSW’s, V’s, and T’s remaining domains have size 3

→ Select SA

Most-Constraining-Variable Heuristic

1) Which variable $X_i$ should be assigned a value next?

Among the variables with the smallest remaining domains (ties with respect to the most-constrained-variable heuristic), select the one that appears in the largest number of constraints on variables not in the current assignment

[Rationale: Increase future elimination of values, to reduce future branching factors]
Map Coloring

- Before any value has been assigned, all variables have a domain of size 3, but SA is involved in more constraints (5) than any other variable.
  - Select SA and assign a value to it (e.g., Blue)

Least-Constraining-Value Heuristic

2) In which order should X’s values be assigned?

Select the value of X that removes the smallest number of values from the domains of those variables which are not in the current assignment.

[Rationale: Since only one value will eventually be assigned to X, pick the least-constraining value first, since it is the most likely not to lead to an invalid assignment]

[Note: Using this heuristic requires performing a forward-checking step for every value, not just for the selected value]
Map Coloring

- Q’s domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

→ So, assign Red to Q
More Advanced Constraint Propagation

- Forward checking can’t discover all possible consequences that could lead to failure
- (Doing this in general would require solving the entire problem, so we shouldn’t expect a free lunch here.)
- AC3 (see textbook) is an advanced algorithm that is a good trade off between efficiency and effectiveness
- But how hard are CSPs, really?

Digression: NP-Hardness

- NP hardness is not an AI topic
- You will not be tested on it explicitly, but
- It’s important for all computer scientists
- Understanding it will deepen your understanding of AI (and other CS) topics
- You will be expected to understand its relevance and use for AI problems
- Eat your vegetables; they’re good for you
P and NP

- P and NP are about decision problems
- P is set of problems that can be solved in polynomial time
- NP is a superset of P
- NP is the set of problems that:
  - Have solutions which can be verified in polynomial time or, equivalently,
  - can be solved by a non-deterministic Turing machine in polynomial time (OK if you don’t know what that means yet)
- Roughly speaking:
  - Problems in P are tractable – can be solved in a reasonable amount of time, and Moore’s law helps
  - Some problems in NP might not be tractable

Scaling
Isn’t P big?

• P includes $O(n)$, $O(n^2)$, $O(n^{10})$, $O(n^{100})$, etc.
• Clearly $O(n^{10})$ isn’t something to be excited about – not practical
• Computer scientists are very clever at making things that are in P efficient
• First algorithms for some problems are often quite expensive, e.g., $O(n^3)$, but research often brings this down

Understanding the class NP

• A class of decision problems (Yes/No)
• Solutions can be verified in polynomial time
• Examples:
  – Graph coloring:
  – Sortedness: [1 2 3 4 5 8 7]
**NP-hardness**

- Many problems in AI are NP-hard (or worse)
- What does this mean?
- NP-hard = as hard as hardest problems in NP
- Identifying a problem as NP hard means:
  - You probably shouldn’t waste time trying to find a polynomial time solution
  - If you find a polynomial time solution, either
    - You have a bug
    - Find a place on your shelf for your Turing award
- NP hardness is a major triumph (and failure) for computer science theory

---

**NP-hardness**

- Why it is a failure:
  - There is a huge class of problems with no known efficient solutions
  - We have failed, as a community, to either find efficient solutions or prove that none exist

- Why it is a triumph:
  - We have a developed a precise language for talking about these problems
  - We have developed sophisticated ways to reason about and categorize the problems we don’t know how to solve efficiently
Hardness vs. Completeness

• For something to be NP-complete, must be in NP
• If something is NP-hard, it could be even harder than the hardest problems in NP

• Theoreticians have shown that if you can solve one NP-complete problem in polynomial time, you can solve ALL NP-complete problems in polynomial time

P=NP?

• Biggest open question in CS

• Can NP-complete problems be solved in poly time?
• Probably not, but nobody has been able to prove it yet

• Many false starts, e.g.:
How challenging is “P=NP?”

• Princeton University CS department
• See: http://www.cs.princeton.edu/general/bricks.php
• Photo from: http://stuckinthebubble.blogspot.com/2009/07/three-interesting-points-on-princeton.html

Hardness of CSPs

• CSPs are known to be NP-complete (for most reasonable formulations of the problem)
• Bad news: Don’t bother trying to find a general, efficient way to solve CSPs
• Good news: Many problems can be solved much faster than the worst (exponential) case practice
• So-so news: Sometimes you just need to run a solver and see what happens
  – You might get an answer quickly
  – You might just wait, and wait, and wait…
CSP Conclusions

• CSPs are a general language for describe a large family of problems
• Could take exponential time to solve in the worst case
• Advanced algorithms exist that try to discover bad choices quickly, effectively reducing the state space
  – Microsoft Solver Foundation
  – CPLEX