Section: Properties of Context-free Languages

Which of the following languages are CFL?

- \( L = \{ a^n b^n c^j \mid 0 < n \leq j \} \)  **NOT CFL**
- \( L = \{ a^n b^j a^n b^j \mid n > 0, j > 0 \} \)  **Not CFL**
- \( L = \{ a^n b^j a^k b^p \mid n + j \leq k + p, n > 0, j > 0, k > 0, p > 0 \} \)  **CFL**
- \( L = \{ a^n b^j a^j b^n \mid n > 0, j > 0 \} \)  **CFL**
Pumping Lemma for Regular Language’s: Let $L$ be a regular language, Then there is a constant $m$ such that $w \in L$, $|w| \geq m$, $w = xyz$ such that

- $|xy| \leq m$
- $|y| \geq 1$
- for all $i \geq 0$, $xy^iz \in L$
Pumping Lemma for CFL’s Let $L$ be any infinite CFL. Then there is a constant $m$ depending only on $L$, such that for every string $w$ in $L$, with $|w| \geq m$, we may partition $w = uvxyz$ such that:

- $|vxy| \leq m$, (limit on size of substring)
- $|vy| \geq 1$, ($v$ and $y$ not both empty)

For all $i \geq 0$, $uv^ixy^iz \in L$

• Proof: (sketch) There is a CFG G s.t. $L = L(G)$.
Consider the parse tree of a long string in $L$.
For any long string, some nonterminal $N$ must appear twice in the path.
Example: Consider
$L = \{a^n b^n c^n : n \geq 1\}$. Show $L$ is not a CFL.

*Proof: (by contradiction)*

Assume $L$ is a CFL and apply the pumping lemma.

Let $m$ be the constant in the pumping lemma and consider
$w = a^m b^m c^m$. Note $|w| \geq m$.

Show there is no division of $w$ into
$uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i xy^i z \in L$ for $i = 0, 1, 2, \ldots$. 
\[ \omega = a^m b^n c^m \]

Case 1: \( V \) nor \( y \) distinct symbols

Case 2: \( v = a \)

Case 3: \( v = b \)

Case 4: \( v = c \)

\( y = a_t^{t_2} \)

\( y = b_t^{t_3} \)

\( u \) \( Vxyz \)
Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i xy^i z$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.
Example Why would we want to recognize a language of the type \( \{a^n b^n c^n : n \geq 1\} \)?
Example: Consider
\[ L = \{a^n b^n c^p : p > n > 0\} \]. Show \( L \) is not a CFL.

Proof: Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider
\[ w = \text{____________} \quad \text{Note } |w| \geq m. \]
Show there is no division of \( w \) into \( uvxyz \) such that \(|vy| \geq 1, |vxy| \leq m, \) and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

Strategy → Choose a tight string to get contradiction easily!

\[ w = a^m b^m c^{m+1} \]
\[ |w| \geq m \]
Strategy → Choose a tight string to get contradiction easily.

\[ \omega = a^m b^m c^{m+1} \]

Case 1: \( \nu \) nor \( \gamma \) distinct symbols
\[ \Rightarrow \nu \) contains \( a's+b's \]
\[ \Rightarrow u v^2 x y^2 z \notin L \]

Case 2: \( \nu \) is all a's

\[ |\omega| \geq m \]
Example: Consider \( L = \{ a^j b^k : k = j^2 \} \). Show \( L \) is not a CFL.

- **Proof:** Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider

\[ w = \text{___________} \]

Show there is no division of \( w \) into \( uvxyz \) such that \( |vy| \geq 1, |vxy| \leq m \), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

**Case 1:** Neither \( v \) nor \( y \) can contain 2 or more distinct symbols. If \( v \) contains \( a \)'s and \( b \)'s, then \( uv^2 xy^2 z \notin L \) since there will be \( b \)'s before \( a \)'s.

Thus, \( v \) and \( y \) can be only \( a \)'s, and \( b \)'s (not mixed).
Example: Consider 
$L = \{ w\bar{w}w : w \in \Sigma^* \}$, $\Sigma = \{a, b\}$, where $\bar{w}$ is the string $w$ with each occurrence of $a$ replaced by $b$ and each occurrence of $b$ replaced by $a$. Show $L$ is not a CFL.

• Proof: Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider 
$w =$ ______________

Show there is no division of $w$ into $uvwxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in L$ for $i = 0, 1, 2, \ldots$. 
Example: Consider $L = \{a^n b^p b^p a^n\}$. $L$ is a CFL. The pumping lemma should apply!

Let $m \geq 4$ be the constant in the pumping lemma. Consider $w = a^m b^m b^m a^m$.

We can break $w$ into $uvxyz$, with:
Chap 8.2 Closure Properties of CFL’s

Theorem CFL’s are closed under union, concatenation, and star-closure.

- Proof:
  
  Given 2 CFG $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$
  
  - Union:
    
    Construct $G_3$ s.t. $L(G_3) = L(G_1) \cup L(G_2)$.
    $G_3 = (V_3, T_3, S_3, P_3)$
– Concatenation:
  Construct $G_3$ s.t. $L(G_3) = L(G_1) \circ L(G_2)$.
  $G_3 = (V_3, T_3, S_3, P_3)$

– Star-Closure
  Construct $G_3$ s.t. $L(G_3) = L(G_1)^*$
  $G_3 = (V_3, T_3, S_3, P_3)$
Theorem CFL’s are NOT closed under intersection and complementation.

• Proof:
  – Intersection:
– Complementation:
Theorem: CFL’s are closed under regular intersection. If $L_1$ is CFL and $L_2$ is regular, then $L_1 \cap L_2$ is CFL.

• Proof: (sketch) We take a NPDA for $L_1$ and a DFA for $L_2$ and construct a NPDA for $L_1 \cap L_2$.

$M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_0, z, F_1)$ is an NPDA such that $L(M_1) = L_1$.

$M_2 = (Q_2, \Sigma, \delta_2, q_0, F_2)$ is a DFA such that $L(M_2) = L_2$.

Example of replacing arcs (NOT a Proof!):
We must formally define \( \delta_3 \). If

then

Must show

if and only if
Questions about CFL:

1. Decide if CFL is empty?

2. Decide if CFL is infinite?
Example: Consider
\[ L = \{ a^{2n} b^{2m} c^n d^m : n, m \geq 0 \} \]. Show \( L \) is not a CFL.

- Proof: Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider
\[ w = a^{2m} b^{2m} c^m d^m \].

Show there is no division of \( w \) into \( uvxyz \) such that \(|vy| \geq 1\), \(|vxy| \leq m\), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

Case 1: Neither \( v \) nor \( y \) can contain 2 or more distinct symbols. If \( v \) contains \( a \)'s and \( b \)'s, then \( uv^2 xy^2 z \notin L \) since there will be \( b \)'s before \( a \)'s.
Thus, \( v \) and \( y \) can be only \( a \)'s, \( b \)'s, \( c \)'s, or \( d \)'s (not mixed).

Case 2: \( v = a^{t_1} \), then \( y = a^{t_2} \) or \( b^{t_3} \) (\(|vxy| \leq m\))
If \( y = a^{t_2} \), then
$u v^2 x y^2 z = a^{2m+t_1+t_2} b^{2m} c^m d^m \notin L$ since $t_1 + t_2 > 0$, the number of $a$’s is not twice the number of $c$’s.

If $y = b^{t_3}$, then

$u v^2 x y^2 z = a^{2m+t_1} b^{2m+t_3} c^m d^m \notin L$ since $t_1 + t_3 > 0$, either the number of $a$’s (denoted $n(a)$) is not twice $n(c)$ or $n(b)$ is not twice $n(d)$.

**Case 3:** $v = b^{t_1}$, then $y = b^{t_2}$ or $c^{t_3}$

If $y = b^{t_2}$, then

$u v^2 x y^2 z = a^{2m} b^{2m+t_1+t_2} c^m d^m \notin L$ since $t_1 + t_2 > 0$, $n(b) > 2* n(d)$.

If $y = c^{t_3}$, then

$u v^2 x y^2 z = a^{2m} b^{2m+t_1} c^m + t_3 d^m \notin L$ since $t_1 + t_3 > 0$, either $n(b) > 2* n(d)$ or $2* n(c) > n(a)$.

**Case 4:** $v = c^{t_1}$, then $y = c^{t_2}$ or $d^{t_3}$

If $y = c^{t_2}$, then

$u v^2 x y^2 z = a^{2m} b^{2m} c^m + t_1 + t_2 d^m \notin L$ since $t_1 + t_2 > 0$, $2* n(c) > n(a)$.

If $y = d^{t_3}$, then
$uv^2xy^2z = a^{2m}b^{2m}c^m + t_1d^{m+t_3} \notin L$ since $t_1 + t_3 > 0$, either $2*n(c) > n(a)$ or $2*n(d) > n(b)$.

**Case 5:** $v = d^{t_1}$, then $y = d^{t_2}$

then $uv^2xy^2z = a^{2m}b^{2m}c^md^{m+t_1+t_2} \notin L$ since $t_1 + t_2 > 0$, $2*n(d) > n(c)$.

Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^ixy^iz$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.