Section: Properties of Regular Languages

Example

$L = \{a^n b a^n \mid n > 0\}$

not regular

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$\Rightarrow L_3 \in \text{class}$
L=\{x \mid x \text{ is a positive even integer}\}

L is closed under

addition? yes
multiplication? yes
subtraction? no \(6-8=-2\)
division?

Closure of Regular Languages

Theorem 4.1 If \(L_1\) and \(L_2\) are regular languages, then

\[L_1 \cup L_2, \quad L_1 \cap L_2, \quad L_1L_2, \quad \overline{L_1}, \quad L_1^*\]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.
$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- Final states in $M$ are nonfinal states in $M'$
- Nonfinal in $M$ are final in $M'$

Show we $L(M') \subseteq L(M)$

$\Rightarrow$ closed under complementation (we construct a DFA for it so $L$ is regular)
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists \text{DFA } M_1 \text{ and } M_2 \text{ s.t.}$

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = Q \times P$

$\delta':$

$\delta'_1((q_i, p_j), a) = (q_k, p_l)$ if

$\delta_1(q_i, a) = q_k \in M_1$

$\delta_2(p_j, a) = p_l \in M_2$

$F' = \{ (q_i, p_j) \in Q' | (q_i \in F_1, p_j \in F_2) \}$

show $w \in L(M') \Rightarrow w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:

\[
\text{\begin{diagram}
\node{1} \arrow{b} \node{2} \\
\node{A} \arrow{a} \node{B} \arrow{a} \node{C} \\
\end{diagram}}
\]

\[
\text{M for intersection of the two DFA}
\]
Regular languages are closed under

reversal $L^R$

difference $L_1 - L_2$

right quotient $L_1 / L_2$

homomorphism $h(L)$
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
\begin{align*}
L_1 &= \{a^*b^* \cup b^*a^*\} \\
L_2 &= \{b^n \mid n \text{ is even, } n > 0\} \\
L_1/L_2 &= \{a^*b^*2\}
\end{align*}
\]
Theorem If \( L_1 \) and \( L_2 \) are regular, then \( L_1/L_2 \) is regular.

Proof (sketch)

\[ \exists\text{ DFA } M=(Q,\Sigma,\delta,q_0,F) \text{ s.t. } L_1 = L(M). \]

Construct DFA \( M'=(Q,\Sigma,\delta,q_0,F') \)

For each state \( i \) do

Make \( i \) the start state (representing \( L'_i \))

\[ \text{if } L_1 \cap L_2 \neq \emptyset \]

\[ \text{put } q_i \text{ in } F' \text{ in } M' \]

QED.
Homomorphism

Def. Let \( \Sigma, \Gamma \) be alphabets. A homomorphism is a function

\[ h: \Sigma \rightarrow \Gamma^* \]

Example:

\[ \Sigma = \{a, b, c\}, \quad \Gamma = \{0, 1\} \]
\[ h(a) = 11 \]
\[ h(b) = 00 \]
\[ h(c) = 0 \]

\[ h(bc) = 000 \]
\[ h(a^b^c^*) = 11(00)^* \]
\[ h(ab^*) = h(a)h(b^*) \]
\[ h(a^b^c^*) = 11(00)^* \]
Questions about regular languages:

L is a regular language.

- Given L, Σ, w ∈ Σ*, is w ∈ L?
  
  Construct FA and test to see if it accepts w.

- Is L empty?
  
  Construct FA. If there is a path from start state to final state, DFS.

- Is L infinite?
  
  DFA is a graph. Is there a cycle?

- Does L₁ = L₂?
  
  Construct L₃ = (L₁ ∩ L₂) ∪ (Σ, o L₂)
  
  If L₃ = ∅ then L₁ = L₂.
Identifying Nonregular Languages

If a language \( L \) is finite, is \( L \) regular?

If \( L \) is infinite, is \( L \) regular?

- \( L_1 = \{a^n b^m | n > 0, m > 0\} = \text{not regular} \)
- \( L_2 = \{a^n b^n | n > 0\} \)
Prove that $L_2 = \{a^n b^n | n > 0\}$ is regular.

- Proof: Suppose $L_2$ is regular.

  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$

  $M$ has a finite no. of states, say $K$ states.

  Consider long string $a^kb \in L_2$

  $K$ states, $K$ a's has to be a loop with a path. $a$ can loop has to have at least one $b$.

  Suppose we start at initial state, traverse path for $a^kb$, but cannot traverse the loop an extra time.

  $\Rightarrow$ we accept $a^{K+3}b^j \notin L_2$, $j > 0$.

  Contradiction, $\Rightarrow L_2$ is not regular.
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^iz \in L \text{ for all } i \geq 0$$
To Use the Pumping Lemma to prove \( L \) is not regular:

- **Proof by Contradiction.**
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) \( L \) satisfies the pumping lemma.
  
  Choose a long string \( w \) in \( L \), \( |w| \geq m \).
  
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m \), \( |y| \geq 1 \) and \( xy^iz \in L \ \forall \ i \geq 0 \).
  
  The pumping lemma does not hold. Contradiction!
  
  \( \Rightarrow \) \( L \) is not regular. QED.
Example $L = \{a^n cb^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  ⇒ the pumping lemma holds.
  Choose $w = a^m c b^m$

Show there is no way to divide into $xyz$ s.t. $P.L.$ holds

$\underbrace{a \cdots a c b \cdots b}_{y \geq 2}$

all partition are in this form

$x = a^k$ $y = a^i$ $z = a^{m-j-k} c b$

$i > 0$ $xyz = a^{m-j} c b$ contradiction

Thus $L$ is not regular
Example $L = \{a^n b^{n+s} c^s \mid n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w =$

  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove L is not regular:

• Proof Outline:
  Assume L is regular.
  Apply closure properties to L and other regular languages, constructing L’ that you know is not regular.
  closure properties \( \Rightarrow \) L’ is regular. Contradiction!
  L is not regular. QED.
Example \( L = \{a^3 b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.
  Define a homomorphism \( h : \Sigma \to \Sigma^* \)
  \[
  h(a) = a \quad h(b) = a \quad h(c) = b
  \]
  \( h(L) = \)
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  Assume $L$ is regular.
Example: \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.