Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

not regular

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
L = \{ x \mid x \text{ is a positive even integer} \}

L is closed under

- addition? yes
- multiplication? yes
- subtraction? no
- division? 

\[ 6 - 8 = -2 \]

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

- \( L_1 \cup L_2 \)
- \( L_1 \cap L_2 \)
- \( L_1 L_2 \)
- \( \overline{L}_1 \)
- \( L_1^* \)

are regular languages.
Proof(sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

Final states in $M$ are nonfinal in $M'$

Nonfinal states in $M$ are final in $M'$

Show $w \epsilon L(M')$ if and only if $\overline{w} \epsilon L$ ($\Rightarrow$ closed under complementation)

(we construct a DFA for it so $L$ is regular)
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1= (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2= (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = Q \times P$

$\delta'$:

$\delta'( ((q_i, p_j), a) ) = (q_k, p_l)$ if

$\delta_1(q_i, a) = q_k \in M_1$

$\delta_2(p_j, a) = p_l \in M_2$

$F' = \{ (q_i, p_j) \in Q' | (q_i \in F_1, p_j \in F_2) \}$

show $w \in L(M') \iff w \in L_1 \land L_2$

$\Rightarrow$ closed under intersection
Example:

M for intersection of the two DFA
Regular languages are closed under

- reversal \( L^R \)
- difference \( L_1 - L_2 \)
- right quotient \( L_1 / L_2 \)
- homomorphism \( h(L) \)
Right quotient

Def: $L_1/L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{ a^*b^* \cup b^*a^* \}$
$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$
$L_1/L_2 = \{ a^*b^* \}$

$L_1 = \{ aaabb, bbbbaa, bbb, aaaaabbb \}$
$L_1/L_2 = \{ aaaa, b, aaaaab, aaib \}$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

Make $i$ the start state (representing $L_i'$)

if $L_1 \cap L_2 \neq \emptyset$

put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$$h(bc) = 000$$

$$h(ab^*) = 11(00)^*$$

$$h(ab^*) = h(a)h(b^*) = 11(00)^*$$
Questions about regular languages:

L is a regular language.

- Given L, \( \Sigma \), \( w \in \Sigma^* \), is \( w \in L \)?
  
  Construct FA and test to see if it accepts \( w \).

- Is L empty?
  
  Construct FA. If there is a path from start state to final state, DFS

- Is L infinite?
  
  DFA is a graph. Is there a cycle?

- Does \( L_1 = L_2 \)?
  
  Construct \( L_3 = (L_1 \cap L_2) \cup (L_1 \cap L_2) \)
  
  If \( L_3 = \emptyset \) then \( L_1 = L_2 \)
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \varepsilon \varepsilon a b b$
- $L_2 = \{a^n b^n | n > 0\}$ not regular
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- **Proof:** Suppose $L_2$ is regular.
  \[ \Rightarrow \exists \text{ DFA } M \text{ that recognizes } L_2 \]

M has a finite no. of states, say K states

Consider long string $a^b b^c \in L_2$

K states, K as's has to be a

loop with a part. But a loop has to have at least one a

Suppose we start at initial state

traverse path for a K b but travel the loop an extra time

\[ \Rightarrow \text{ we accept } a^{K+3} b^j L_2, j > 0 \]

Contradiction, \( \Rightarrow L_2 \) is not regular
Pumping Lemma: Let $L$ be an infinite regular language. There exists a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$
|xy| \leq m \\
|y| \geq 1 \\
x y^i z \in L \text{ for all } i \geq 0
$$
To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.
  Assume L is regular.
  ⇒ L satisfies the pumping lemma.
  Choose a long string $w$ in L, $|w| \geq m$.
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  The pumping lemma does not hold. Contradiction!
  ⇒ L is not regular. QED.
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w = a^m c b^m$

  Show there is no way to divide $w$ into $xyz$ s.t. $P.L.$ holds

  All partition are in this form:

  $x = a^k$, $y = a^j$, $z = a^{m-j-k} b^m$

  $i = 0$  $xy^iz = a^m c b^{m+k}$ is a contradiction

  Thus $L$ is not regular.
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w = a^m b^{m+s} c^s$

  So the partition is:

  only way to divide $xyz$
  
  $x = a^k$, $y = a^j$, $z = a^{m-k-j} b^{m+s} c^s$
  
  $i = 2$, $xyyz = a^m b^{m+s} c^s \notin L$

  no. $a$'s + $c$'s $\neq$ no. $b$'s too many $a$'s

  contradiction

  $\Rightarrow L$ is not regular
Example $\Sigma = \{a, b\}$, 
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w = a^{m+1} b^m$

So the partition is:

- only one way

  $X = a^k \quad y = a^j \quad Z = a^m b^m$

  $i = 0 \quad x y^i z = x z = a^{m+1} b^m \notin L$

- no. of $a$'s $\leq$ no. of $b$'s

  Contradiction

$\Rightarrow L$ is not regular
Example \( L = \{ a^3 b^n c^{n-3} | n > 3 \} \)
(shown in detail on handout)

\( L \) is not regular.
To Use Closure Properties to prove $L$ is not regular:

- Proof Outline:
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  closure properties $\Rightarrow L'$ is regular. Contradiction!
  $L$ is not regular. QED.
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  
  Assume $L$ is regular.
  
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  
  $h(a) = a$  $h(b) = a$  $h(c) = b$
  
  $h(L) = \{a^n b^{n-3} | n > 3\}$
  
  $= \{a^{n+3} b^{n-3} | n > 3\}$

$\{b, bb, bbb, bbbb, \}$

$L' = h(L)$  $\subseteq b^3 = \{a^{n+3} b^{n+3} | n > 3\}$

$L'' = \{a b, aabb, a^3 b^3, a^4 b^4, a^5 b^5, a^6 b^6\}$

$L''' = \{a^6 b^6 | n > 3\}$ regular

Contradiction!  
$
\Rightarrow L$ is not regular.
Example $L=\{a^nb^ma^m|m \geq 0, n \geq 0\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.

  $L_1 = \exists b^*a^*b^3$ is regular

  $L_2 = L_1 \cap L = \{b^n \alpha^n | n > 0\}$

  $h(a) = b$, $h(b) = a$

  $h(L_2) = \exists a^n b^n | n > 0$ is not regular

  Contradiction

  $\Rightarrow L$ is not regular
Example: \( L_1 = \{a^n b^n a^n | n > 0\} \)

\( L_1 \) is not regular.

\[
L_2 = \{a^*b^* \} \text{ is regular}
\]

\[
L_3 = L_1 \setminus L_2 = \{a^n b^n a^p | n > 0, 0 \leq p \leq n\}
\]

\[
L_4 = L_3 \cap \{a^*b^* \} = \{a^n b^n | n \geq 0\}
\]

Contradiction!

\[ \Rightarrow L \) is not regular \]