Problem 1 (Communications). (15 points) Alice has a unit vector $x \in \mathbb{R}^n$, Bob has a unit vector $y \in \mathbb{R}^n$. The two vectors either has an angle that is at most $\alpha$, or at least $\beta$ ($0 \leq \alpha < \beta \leq \pi$). They also share public bits of randomness.

Show that they can communicate $O(1/(\beta - \alpha)^2)$ bits, and determine whether the angle is at most $\alpha$ or at least $\beta$ with success probability at least $3/4$.

Note that communicating a real number (with reasonable precision) counts as $O(\log n)$ bits and is not allowed. (Hint: Use hyperplanes.)

Problem 2 (Preserving Inner-Products). Recall that in order to prove Johnson-Lindenstrauss Lemma, we used the following concentration result:

**Lemma 1.** Let $v$ be a vector in $m$ dimensions. Suppose $A \in \mathbb{R}^{d \times m}$ is a random matrix whose entries are i.i.d. from $N(0, 1)$. For any $\eta > 0$, when $d > C \frac{\log(1/\eta)}{\epsilon^2}$ for a large enough universal constant $C$, we have with probability at least $1 - \eta$

$$ (1 - \epsilon)\|v\|_2^2 \leq \frac{1}{d} \|Av\|_2^2 \leq (1 + \epsilon)\|v\|_2^2. $$

We are going to prove variants of Johnson-Lindenstrauss Lemma using similar ideas.

Throughout the problem, we are given $n$ vectors $v_1, v_2, ..., v_n \in \mathbb{R}^m$. Matrix $A$ is defined the same as the Lemma above with $d = O \left( \frac{\log n}{\epsilon^2} \right)$ dimensions. (possibly with a larger constant compared to the Lemma, and possibly different for each sub-problem)

(a) (5 points) Show that for any pair $(i, j)$,

$$ (1 - \epsilon)\|v_i \pm v_j\|^2 \leq \frac{1}{d} \|Av_i \pm Av_j\|^2 \leq (1 + \epsilon)\|v_i \pm v_j\|^2 $$

(b) (10 points) Show that for any pair $(i, j)$,

$$ \langle v_i, v_j \rangle - \epsilon(\|v_i\|^2 + \|v_j\|^2) \leq \frac{1}{d} \langle Av_i, Av_j \rangle \leq \langle v_i, v_j \rangle + \epsilon(\|v_i\|^2 + \|v_j\|^2). $$

(Hint: Express inner-products $\langle v_i, v_j \rangle$ with $\|v_i + v_j\|$ and $\|v_i - v_j\|$.)
(c) (10 points) Show that for any pair \((i,j)\),
\[
\langle v_i, v_j \rangle - \epsilon \|v_i\| \|v_j\| \leq \frac{1}{\delta} \langle Av_i, Av_j \rangle \leq \langle v_i, v_j \rangle + \epsilon \|v_i\| \|v_j\|. 
\]

(Hint: Think about when does \(\|v_i\| \|v_j\|\) differ significantly from \((\|v_i\|^2 + \|v_j\|^2)\).)

**Problem 3** (Graph sparsifier). Let \(C_n\) be a cycle with \(n\) vertices. That is, the graph has \(n\) edges: there is an edge between \((i, i+1)\) for \(i = 1, 2, ..., n-1\) and there is also an edge \((n, 1)\).

Let \(P_n\) be a path with \(n\) vertices. That is, the graph has \(n-1\) edges, there is an edge between \((i, i+1)\) for \(i = 1, 2, ..., n-1\).

(a) (10 points) Let \(w_{C}(S, \bar{S})\) be the capacity of the cut in \(C_n\), and \(w_{P}(S, \bar{S})\) be the capacity of the cut in \(P_n\). Show that for every cut \((S, \bar{S})\),
\[
w_{P}(S, \bar{S}) \leq w_{C}(S, \bar{S}) \leq 2w_{P}(S, \bar{S}).
\]

(b) (10 points) Let \(L_{C}\) be the Laplacian matrix of \(C\), \(L_{P}\) be the Laplacian matrix of \(P\), show that there exists a vector \(x \in \mathbb{R}^n\) such that
\[
x^\top L_{C}x \geq nx^\top L_{P}x.
\]

(So, \(P_n\) is a cut sparsifier of \(C_n\), but not a spectral sparsifier.)