This recitation is to review the material from sections 1.6-1.8 and sections 2.1-2.2 in the course textbook as covered in lectures 2 and 3.

1. Prove that for any real numbers $x$ and $y$, $|x + y| \leq |x| + |y|$. Avoid "obvious" facts about absolute value, and use its definition to derive any facts you need:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Solution: First, let’s prove a useful fact: for any $x$, we have $x \leq |x|$. To see this, consider an arbitrary real number $x$. There are two cases for $x$: either $x \geq 0$, or $x < 0$. In the former case, then $x = |x|$ by definition which directly implies (the weaker statement) $x \leq |x|$. In the latter case we have $x < 0 < -x$, and $-x = |x|$ by definition, so $x < |x|$ which directly implies (the weaker statement) $x \leq |x|$. Thus, in either case, we conclude $x \leq |x|$. Since these cases are exhaustive, we conclude $x \leq |x|$ for any real $x$.

Similarly, we can prove the fact that $-x \leq |x|$ for any real $x$. Let $x$ be an arbitrary real number. If $x \geq 0$, then $-x \leq x = |x|$ as desired, and if $x < 0$, then $-x = |x|$ so $-x \leq |x|$. Again, these cases are exhaustive, so we conclude $-x \leq |x|$ for any real $x$.

With these facts in hand, we prove the claim set out. Let $x$ be an arbitrary real number. Applying the definition of absolute value, we have two cases for $x + y$:

(a) Case $x + y \geq 0$: Then $|x + y| = x + y$ by definition. By the fact above, we have $x + y \leq |x| + |y|$ as desired.

(b) Case $x + y < 0$: Then $|x + y| = -(x + y)$ by definition. Rewriting this term, we have $(−x) + (−y)$. By the second fact above, we have $(−x) + (−y) \leq |x| + |y|$ as desired.

Since the proposition is true in each case and the cases are exhaustive, the proposition is true. □

2. Prove that for any positive integer $n$, $n$ is divisible by 5 if and only if the least significant (rightmost) digit of $n$ is 0 or 5.

Discussion: Since last discussion, there have been questions regarding what is sufficient to prove a non-negative real number $n$ is divisible by a positive integer $k$. A useful fact that you can use in your own proofs is that for any non-negative integer $k$ and non-negative integers $a$ and $b$, we have $ka + b$ is divisible by $k$ if and only if $b$ is divisible by $k$.

For example, recall the term $4k^2 + 4k + 1$ for some positive integer $k$ from the previous recitation. To prove this quantity is not divisible by 4, we can use the fact above: $4k^2 + 4k + 1$ can be written as $4(k^2 + k) + 1$, and since $1 < 4$, we have that $4(k^2 + k) + 1$ is not divisible by 4.

Solution (By chain of if and only ifs): Let $n$ be an arbitrary positive integer, and let $y$ be the least significant (rightmost) digit of $n$. Then $y$ is a non-negative integer between 0 and 9. We know $n – y$ is divisible by 10, so $n – y = 10x$ for some non-negative integer $x$, and then $n$ can be expressed as $10x + y$. It follows that $n$ is divisible by 5 if and only if $10x + y$ is divisible by 5. Since $10 = 5 \cdot 2$, we have $n = 5 \cdot 2x + y$. By the fact above, we have $5 \cdot 2x + y$ if and only if $y$ is divisible by 5. Since $0 \leq y < 10$, $y$ is divisible by 5 if and only if $y$ is 0 or 5, which completes the proof. □
Solution (By proof of both implications): Let \( n \) be an arbitrary positive integer that is divisible by 5. Then \( n = 10x + y = 5 \cdot 2x + y \) for some positive integer \( x \) and non-negative integer \( y < 10 \). By the fact above, \( y \) is divisible by 5. Since \( 0 \leq y < 10 \), we have \( y \) is either 0 or 5 as desired. This proves that if \( n \) is divisible by 5, then \( y \) is either 0 or 5.

Now assume the last digit \( y \) of \( n \) is 0 or 5. In any case, \( n = 10x + y \) for some non-negative integer \( x \). If \( y = 0 \), then \( n = 10x \) which is divisible by 5. Otherwise, \( y = 5 \) and \( n = 10x + 5 = 5(2x + 1) \) which is divisible by 5. Since these cases are exhaustive, we have that if the last digit of \( n \) is either 0 or 5, then \( n \) is divisible by 5.

Since we have proven both implications, the given proposition is true.

3. Prove that for any positive integer \( n \), \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).

Solution: We will prove the proposition true using the well ordering principle. First, define \( S \) to be the set of all counterexamples to the proposition above; that is, let \( S \) be the set of all positive integers for which

\[
\sum_{i=1}^{n} i \neq \frac{n(n+1)}{2}.
\]

Assume for sake of contradiction that the proposition does not hold for all \( n \), so \( S \) is non-empty. Since \( S \) is a non-empty set of positive integers, the well ordering principle implies there is a smallest integer in \( S \). Let \( k \) be the smallest integer in \( S \). Since \( k \) is in \( S \), we have

\[
\sum_{i=1}^{k} i \neq \frac{k(k+1)}{2}.
\]

Subtracting \( k \) from each side yields:

\[
\sum_{i=1}^{k} i - k \neq \frac{k(k+1)}{2} - k
\]

\[
\Rightarrow \sum_{i=1}^{k-1} i \neq \frac{k}{2} ((k+1) - 2)
\]

\[
\Rightarrow \sum_{i=1}^{k-1} i \neq \frac{(k-1)k}{2}
\]

If \( k - 1 \) is a positive\(^1\) integer, then \( k - 1 \) is a counterexample and is in \( S \). However, \( k - 1 < k \), so \( k - 1 \) is a smaller counterexample than \( k \), which is a contradiction.

In the case that \( k - 1 \) is not positive, then we cannot use the same argument as above. However, in this case, it must be that \( k = 1 \) since \( k \) is positive. Then we observe that the proposition does hold for \( k \); indeed, \( \sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} \). This implies \( k \) is not in \( S \), but \( k \) is the smallest integer in \( S \) which is a contradiction.

These cases are exhaustive, and in each case we reached a contradiction. Thus, we conclude our assumption that \( S \) is non-empty is false, and thus the given proposition holds for all positive integers \( n \).

\(^1\)Recall that the proposition is only stated for positive integers.
4. Consider the following scenario. There are \( n \) college basketball teams, and every team will play exactly one game against every other team. In each game, exactly one team wins (there are no ties). If team \( a \) wins in a game against team \( b \), we say \( a \) beats \( b \). A cycle of length \( k \) is a sequence \( t_1, t_2, \ldots, t_k \) such that \( t_i \) beats \( t_{i+1} \) for all \( i < k \), and \( t_k \) beats \( t_1 \). In other words, \( t_1 \) beats \( t_2 \), \( t_2 \) beats \( t_3 \), etc., and additionally \( t_k \) beats \( t_1 \). Notice any cycle has length at least three.

Show that for any \( n \) and any outcomes of the games, if there is a cycle of length at least three, then there is a cycle of length exactly three.

**Solution:** Let \( n \) be an arbitrary positive integer \( n \), and consider arbitrary outcomes of all games. Let \( S \) be the set of lengths of cycles for those outcomes; that is, integer \( k \) is contained in \( S \) if and only if there is a cycle of length \( k \). Since \( S \) contains only positive integers, the well ordering principle implies there is a smallest integer \( k \) in \( S \). If \( k = 3 \), then there is a cycle of length 3, so the proposition holds in this case. Otherwise, \( k > 3 \) (since there are no cycles shorter than length 3). Since \( k \) is in \( S \), there is some cycle of length \( k \): \( t_1, t_2, \ldots, t_k \). There are two cases for the outcome of the game played by teams \( t_1 \) and \( t_3 \):

(a) Case \( t_1 \) beat \( t_3 \): Then \( t_1, t_3, t_4, \ldots, t_k \) is a cycle of length \( k - 1 \), so \( k - 1 \) is in \( S \). However, \( k - 1 < k \) and \( k \) is the smallest integer in \( S \), which is a contradiction. Thus, it must not be the case that \( t_1 \) beat \( t_3 \).

(b) Case \( t_3 \) beat \( t_1 \): Then \( t_1, t_2, t_3 \) is a cycle of length 3.

Since the proposition holds in all cases and the cases are exhaustive, the proposition holds true.

**Note:** The proof above does not follow the standard "WOP template" as described in section 2.2 of the textbook, but it does invoke the well ordered principle.