Recall the properties of relations on sets (where $R$ is the relation on set $A$):

- **Reflexive**: $\forall a \in A. aRa$.
- **Irreflexive**: $\forall a \in A. \neg (aRa)$.
- **Transitive**: $\forall a, b, c \in A. aRb \land bRc \rightarrow aRc$.
- **Symmetric**: $\forall a, b \in A. aRb \rightarrow bRa$.
- **Asymmetric**: $\forall a, b \in A. aRb \rightarrow \neg (bRa)$.
- **Antisymmetric**: $\forall a, b \in A. aRb \land bRa \rightarrow a = b$.

1. Prove by ordinary induction that, for any natural number $n$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

**Solution**: Let predicate $P(n)$ be “$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$” for any $n \in \mathbb{N}$.

**Base case**: Since both $\sum_{i=0}^{1} 2^i$ and $2^2 - 1$ equal 3, $P(1)$ is true.

**Inductive hypothesis**: Let $n$ be an arbitrary natural number. Assume $P(n)$ is true.

**Inductive step**: Since $P(n)$ is true, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ by the inductive hypothesis. It follows that:

\[
\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i
\]

\[
= 2^{n+1} + (2^{n+1} - 1) \quad \text{(definition of $\sum$)}
\]

\[
= 2(2^{n+1}) - 1
\]

\[
= 2^{n+2} - 1
\]

Thus, $P(n + 1)$ is true assuming $P(n)$ for any $n \in \mathbb{N}$.

Since we have shown $P(1)$ is true and $P(n) \rightarrow P(n + 1)$ for any $n \in \mathbb{N}$, we conclude $P(n)$ is true for all $n \in \mathbb{N}$ by ordinary induction. 

2. Prove by ordinary induction that, for any natural number $n$, $3^{n+2} + 2^{4n+2}$ is divisible by 13. [Hint: Does this look familiar? How can you leverage your proof that used the WOP?]

**Solution**: Let predicate $P(n)$ be “$3^{n+2} + 2^{4n+2}$ is divisible by 13” for any $n \in \mathbb{N}$.

**Base case**: Since $3^{1+2} + 2^{4+2} = 91 = 13(7)$ is clearly divisible by 13, $P(1)$ is true.

**Inductive hypothesis**: Let $n$ be an arbitrary natural number. Assume $P(n)$ is true.
Inductive case: Since $P(n)$ is true, $3^{n+2} + 2^{4n+2} = 13k$ for some $k \in \mathbb{N}$. It follows that:

$$3^{(n+1)+2} + 2^{4(n+1)+2} = 3^{(n+3)} + 2^{4n+6}$$
$$= 3(3^{n+2}) + 2^4(2^{4n+2})$$
$$= 3(3^{n+2}) + 16(2^{4n+2})$$
$$= 3(3^{n+2}) + 3(2^{4n+2}) + 13(2^{4n+2})$$
$$= 3(3^{n+2}) + 2^{4n+2} + 13(2^{4n+2})$$
$$= 3(13k) + 13(2^{4n+2})$$

(inductive hypothesis)

Thus, $3^{(n+1)+2} + 2^{4(n+1)+2}$ is divisible by 13, so $P(n + 1)$ is true assuming $P(n)$ for any $n \in \mathbb{N}$.

Since we have shown $P(1)$ is true and $P(n) \rightarrow P(n + 1)$ for any $n \in \mathbb{N}$, we conclude $P(n)$ is true for all $n \in \mathbb{N}$ by ordinary induction.

3. Consider the relation $R$ on $\{1, 2, 3, 4, 5\}$ is an equivalence relation:

$$R = \{(1, 1), (1, 4), (4, 1), (4, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

(a) Verify $R$ is an equivalence relation.

(b) What is $[3]$?

(c) What is the partition induced by $R$?

Solution (a): We need to verify that $R$ is reflexive, transitive, and symmetric to be an equivalence relation. It is easy to check that $R$ is reflexive by observing $aRa$ for all $a \in A$. To see that $R$ is symmetric, we see that for each $(a, b) \in R$, $(b, a)$ is also in $R$. To see that $R$ is transitive, we need to show for any $a, b, c \in A$, if $aRb$ and $bRc$, then $aRc$. The implication is vacuously true for any $a, b, c$ such that $\neg(aRb)$ or $\neg(bRc)$, so now we only need to show the implication holds for $a, b, c \in A$ where the antecedent is true. For example, if $b = 1$, this is to verify that the following four propositions are true:

$$1R1 \land 1R1 \rightarrow 1R1$$
$$1R1 \land 1R4 \rightarrow 1R4$$
$$4R1 \land 1R1 \rightarrow 4R1$$
$$4R1 \land 1R4 \rightarrow 4R4$$

After listing the implications where the antecedent is true for each $b \in A$ (as done above for only $b = 1$), we verify that all implications are true, and thus $R$ is transitive.


Solution (c): The partition induced by an equivalence relation $R$ on a set $A$ is the set of (distinct) equivalence classes of the elements in $A$. Thus, the partition for this equivalence relation $R$ is $\{\{5\}, \{1, 4\}, \{2, 3\}\}$. 


4. Consider the relation \( R \) on set \( A = \{ n \in \mathbb{Z} \mid 1 \leq n \leq 10 \} \):

\[
R = \{ (x, y) \in A \times A \mid x = y \lor (x \text{ is odd } \land x < y) \}
\]

(a) Verify that \( R \) is a weak partial order.

(b) What is the size of the largest chain in \( R \)?

(c) What is the size of the largest antichain in \( R \)?

(d) At least how many chains must any chain decomposition of \( R \) have?

Solution (a): For \( R \) to be a weak partial order, \( R \) must be transitive and antisymmetric. To see the latter, suppose \( R \) is not antisymmetric for sake of contradiction. Then there must be \( a, b \in A \) such that \( aRb, bRa, \) and \( a \neq b \). Without loss of generality, \( a < b \). Since \( a \neq b \), either \( a < b \) or \( b < a \), so either \( \neg(aRb) \) or \( \neg(bRa) \), a contradiction.

To see that \( R \) is transitive, consider any \( a, b, c \in A \) such that \( aRb \) and \( bRc \). There are three cases to consider.

(a) Case \( a = b \): Then \( bRc \rightarrow aRc \).

(b) Case \( a \neq b \) and \( b = c \): Then \( aRb \rightarrow aRc \).

(c) Case \( a \neq b \) and \( b \neq c \): Since \( aRb \) and \( a \neq b \), \( a < b \) and \( a \) is odd. Similarly, since \( bRc \) and \( b \neq c \), \( b < c \) and \( b \) is odd. \( a < b \) and \( b < c \) implies \( a < c \). Since \( a \) is odd, we have \( aRc \).

\[\blacksquare\]

Solution (b): There is a chain of length 6, \( C = \{1, 3, 5, 7, 9, 10\} \). See that for any two distinct odd numbers \( a, b \in A \), \( aRb \) or \( bRa \), so they are comparable. Then see for any odd number \( a \in A, aR(10) \). Indeed, \( a \) is odd and \( a < 10 \) since all numbers in \( A \) are at most 10, so \( aR(10) \). Thus, \( C \) is indeed a chain.

To prove that there is no longer chain, observe that \( A \) contains five odd numbers and five even numbers. Thus, any subset of \( A \) of size more than six has at least two even numbers. Any two distinct even numbers \( a, b \in A \) are incomparable; indeed, neither \( aRb \) nor \( bRa \) since \( a \neq b \) and neither are odd. Thus, any chain has at most one even number. Any chain has at most all five odd numbers, so a largest chain is at most size six. \( C \) has size six, so \( C \) is a largest chain.

\[\blacksquare\]

Solution (c): There are two antichains of length 5: \( D_1 = \{2, 4, 6, 8, 9\} \) and \( D_2 = \{2, 4, 6, 8, 10\} \). As mentioned above, any two distinct even numbers are incomparable, so these subsets are indeed antichains.

To prove that there is no longer antichain, suppose for sake of contradiction that \( D \) is an antichain such that \( |D| > 5 \). First, see that \( D \) must contain at least one odd number since there are only five even numbers in \( A \). Then see \( D \) has at most one odd number since any two distinct odd numbers are comparable as mentioned above. Since \( D \) contains all even numbers, it contains 10, but 10 is comparable with every odd number in \( A \), and thus is comparable with the odd number in \( D \), a contradiction. We conclude that \( |D| \leq 5 \), and since \( |D_1| = |D_2| = 5 \), \( D_1 \) and \( D_2 \) above are longest antichains of \( R \).

\[\blacksquare\]
Solution (d): Every element of an antichain must be in its own chain. As shown in part c, the longest antichain has size five, so any chain decomposition has at least five chains. (From lecture 10 we actually know the smallest chain decomposition has exactly five chains; we will prove this fact in a later lecture.) ■

5. Let $A$ be a set and let $R$ be a relation on $A$.

(a) Prove or disprove that there exists an equivalence relation $S$ on $A$ such that $R \subseteq S$.
(b) Prove or disprove that there exists a strong partial order $S$ on $A$ such that $S \subseteq R$.

Solution (a): The statement is true. To see this, let $S = A \times A$. Clearly $R \subseteq S$, and from the previous recitation, we have that $S$ is reflexive, transitive, and symmetric, so $S$ is an equivalence relation on $A$. ■

Solution (b): The statement is true. To see this, let $S = \emptyset$. Clearly $S \subseteq R$, and from the previous recitation, we have that $S$ is transitive and asymmetric, so $S$ is a strong partial order on $A$. ■