- NP-hardness
  - Problem $A$ is NP-hard if for any problem $B$ in NP, $B$ can be reduced to $A$ in polynomial time.
  
  $B \leq A$

- If both $A$ and $B$ are complete:
  1. $A$ and $B$ are in NP
  2. $A$ and $B$ are NP-hard

Know $A \leq B, B \leq A$

- "Chain" property for reductions

$A \leq B, B \leq C \implies A \leq C$

- Reductions

1. $INDSET \rightarrow CLIQUE$

   - Instance of $INDSET$: Graph $G$, number $k$
   - Instance of $CLIQUE$: Graph $G$, number $k'$

   Reduction $(G, k) \rightarrow (G', k')$

   - $INDSET \rightarrow CLIQUE$

   Idea: if a set $S$ is an IND-SET of $G$
   
   \[\Rightarrow S \text{ is a CLIQUE in } G'\]

   \[\Rightarrow G' \text{ "negative" of } G\]
2. 3-SAT → IND-SET

\[ x_1 = 1 \quad x_2 = 0 \quad x_5 = 0 \]

instance of 3-SAT \((x_1 \lor x_2 \lor x_4) \land (x_2 \lor x_3 \lor x_5)\)

instance of IND-SET Graph \(G_1\), number \(k\).

- step 1 3-SAT → IND-SET

for every variable \(x_i\), construct two vertices

\[ U_i, V_i \]

connect \((U_i, V_i)\)

hope: \(x_i = \text{true} \quad U_i\) is in ind.set

\(x_i = \text{false} \quad V_i\) is

for every clause \(C_j = (x_i \lor x_2 \lor \overline{x}_i)\)

construct a gadget

\[ W_{ij}, W_{i2}, W_{i3} \]

intuition: clause is satisfied, can select one of

the 3

is not sat.: cannot select any of the 3

Size of ind. set \(K = n + m\)

\(X_i = \text{true} \to U_i\)

\(\sim X_i \to U_i \land \sim U_i\)

2. if 3-SAT is satisfiable, there is an IND-SET of size \(K = n + m\).
\[ x_i = \text{true} \rightarrow u_i \]
\[ x_i = \text{false} \rightarrow \overline{u}_i \]

Each clause \( \rightarrow \) select the \( w \) vertex that corresponds to one of the satisfied literals.

3. If there is an ind-\( \text{set} \) of size \( k \), 3-SAT is satisfiable by construction can only select \( k = \sum_{i=1}^{n} m_{i} \) vertex from each variable gadget clause gadget

\( \sum_{i=1}^{n} m_{i} \)

We must have exactly one vertex selected in each gadget.

\[ x_i = \text{true} \leftarrow u_i \]
\[ x_i = \text{false} \leftarrow \overline{u}_i \]