- Shortest path with negative edge length

- Arbitrage example
  
  \((u_1, u_2, \ldots, u_k)\)

  1 unit of \(u_1\), \(C(u_i, u_{i+1})\): 1 unit of \(u_{i+1}\) = \(C(u_i, u_{i+1})\) unit of \(u_i\)

  If we define \(W(u, v) = \log C(u, v)\) then

  \[
  \frac{\log \frac{1}{C(u, u_1)} \cdot \frac{1}{C(u_1, u_2)} \cdots \frac{1}{C(u_{k-1}, u_k)}}{\text{length of path}} = \sum_{i=1}^{k-1} W(u_i, u_{i+1})
  \]

- Shortest path and negative cycles

  Claim: if \(s\) can reach a negative cycle \((u_1, u_2, \ldots, u_k)\) then the shortest path from \(s\) to any \(u_i\) is not defined.

- Bellman Ford

  \[d(u, v): \text{length of shortest path from } s \text{ to } v \text{ using at most } 0 \text{ edges}\]

  \[d(u, 1): \text{length of shortest path using at most } 1 \text{ edge}\]

  \[d(u, 2): \text{length of shortest path using at most } 2 \text{ edges}\]
Theorem: If the graph does not have any negative cycle, then $d(u, n-1)$ will be the shortest path distance from $s$ to $u$.

Also, $d(u, n) = d(u, n-1)$ for any vertex $u$.

On the other hand, if there is a negative cycle reachable from $s$, there exists vertex $u$ s.t. $d(u, n) < d(u, n-1)$

Proof:

1. Prove that $d(u, i)$ is indeed length of shortest path from $s$ to $u$ using at most $i$ edges.
2. Claim: shortest path from $s$ to $u$ has length $\leq n-1$

   Idea: show path will visit each vertex at most once.

3. 
\[ w(u_1, u_2) + w(u_2, u_3) + \ldots + w(u_{k-1}, u_k) + w(u_k, u_1) < 0 \]

\[ d(u_{i+1}, n) \leq d(u_i, n-1) + w(u_i, u_{i+1}) \]

\[ d(u_1, n) \leq d(u_k, n-1) + w(u_k, 1) \]

Take the sum:

\[ \sum_{i=1}^{n} d(u_i, n) \leq \sum_{i=1}^{n} d(u_i, n-1) + \text{length of cycle} \]

\[ < \sum_{i=1}^{n} d(u_i, n-1) < 0 \]

- Running time: \( O(nm) \)

- Minimum spanning tree

- Spanning tree

A spanning tree of a graph \( G = (V, E) \) is a subset of \( n-1 \) edges in \( E \), such that all pairs of vertices are connected by these edges.

Properties for MST:

Subtrees of MST are also MST.
- Cuts in graphs

Construct a cut: choose a set of vertices $S$

$C(S, \overline{S}) = \{ (u,v) \in E \mid u \in S, v \in \overline{S} \}$

If $(u,v) \in E$, $u \in S$, $v \in \overline{S}$ say edge $(u,v)$ "crosses" the cut $(S, \overline{S})$

- For any spanning tree $T$, any cut $(S, \overline{S})$

There must be at least one edge $(u,v)$ in both $T$ and $(S, \overline{S})$