- Random variable: $X$ is a random variable if its value depends on some random events.

$\Pr[X = v]$ denotes the probability that $X$ is equal to $v$.

E.g., if $X \sim \text{coin}$, $X = \begin{cases} 0 \quad \text{w.p. } \frac{1}{2} \\
1 \quad \text{w.p. } \frac{1}{2} 
\end{cases}$

$X = \begin{cases} \frac{1}{2} \quad \text{w.p. } \frac{1}{6} \\
\frac{3}{2} \quad \text{w.p. } \frac{1}{6} 
\end{cases}$

- $X = v$, $X > v$, $X < v$ are called "events" $X \in \mathcal{S}$

- Joint probability: Probability that two events happen at the same time.

$\Pr[X = i, Y = j]$

Independence if $X$, $Y$ are independent:

$\Pr[X = i, Y = j] = \Pr[X = i] \Pr[Y = j]$

- Conditional probability

$\Pr[X = i \mid Y = j]$: Probability of $X = i$, given that we already know $Y = j$.

If $X$, $Y$ are independent:

$\Pr[X = i \mid Y = j] = \Pr[X = i]$

Bayes rule

$$\Pr[X = i \mid Y = j] = \frac{\Pr[X = i, Y = j]}{\Pr[Y = j]} = \frac{\Pr[Y = j \mid X = i] \cdot \Pr[X = i]}{\Pr[Y = j]}$$

- Expectation

$$E[X] = \sum_{x \in \mathcal{S}} x \cdot \Pr[X = x]$$
- expectation
\[ E[X] = \sum_i \Pr[X = i] \times i \]

linearity of expectation
\[ E[X + Y] = E[X] + E[Y] \]
true even if \( X \) and \( Y \) are not independent.

- conditional expectation
\[ E[X \mid Y = j] = \sum_i \Pr[X = i \mid Y = j] \times i \]
\[ E[X] = \sum_j E[X \mid Y = j] \times Pr[Y = j] \]
\[ = \sum_j E[X \mid Y = j] \times \Pr[Y = j] \]

- Quick Sort

- Let \( X_n \) be running time of quicksort on array of length \( n \)
\( X_n \) is a random variable

- Event that the first pivot number is equal to the \( i \)-th smallest number in the array

\[ E[X_n] = \sum_{i=1}^{n} E[X_n \mid \text{pivot} = i] \times \Pr[\text{pivot} = i] \]

for any \( i \), \( \Pr[\text{pivot} = i] = \frac{1}{n} \)
\[ E[X_n \mid \text{pivot} = i] = E[X_{i-1} + A \cdot n + X_{n-i} \mid \text{pivot} = i] \]

\( A \) time takes to sort
\( n \) time takes to split the array
\( n \) time takes to sort the
\[ \sum_{i=1}^{n} A \cdot n \cdot \log_{2} n = A \cdot \sum_{i=1}^{n} n \cdot \log_{2} n \]

\[ = A \cdot n \cdot \log_{2} n \cdot \sum_{i=1}^{n} 1 \]

\[ \leq A \cdot n \cdot \frac{2 \cdot \log_{2} n}{\log_{2} 2} \]

\[ \leq A \cdot n \cdot \log_{2} n \]

\[ \leq \frac{2 \cdot \log_{2} n}{\log_{2} 2} \]

\[ = \frac{2 \cdot \log_{2} n}{1} \]

\[ \leq 2 \cdot \log_{2} n \]

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\[ \leq C \cdot n \log_2 n \quad \text{true when } C \geq 4A \]

by induction can choose \( C = 4A \)

\[ \mathbb{E}[X_n] \leq 4A n \log_2 n \]