1 Linear Program

Most optimization problems are defined by three major components:

1. Set of Variables (Parameters)
2. Constraints
3. Objective

In linear programming, the components are defined as:

1. Variables: \( n \) real numbers, \( x_1, x_2, \ldots, x_n \in \mathbb{R} \).
2. Constraints: a set of linear inequalities. For instance, these are a set of valid linear inequalities:
   (a) \( 2 \times x_1 \geq x_2 - x_3 \)
   (b) \( x_1 \leq x_5 + 10 \)
   (c) \( c \times x_1 \leq x_2 \), given \( c \) is a known constant

   These are not valid linear inequalities:
   (a) \( x_1 \times x_2 \leq 1 \)
   (b) \( \log(x_1) + \log(x_3) \leq 3 \)
   (c) \( \log(x_1) \leq 10 \)
3. Objective: Linear function over the variables.

An example of the linear program is illustrated as follows:

1. Variables: \( x, y \)
2. Constraints:

\[
\begin{align*}
x & \geq 0 \\
y & \geq 0 \\
x + y & \leq 1
\end{align*}
\]
3. Objective: \( \text{max}(2 \times x + y) \)
2 Solutions to linear program

Solution: An assignment of the variables
Feasible Solution: A solution that satisfies all constraints. For instance, for the example linear program problem above, some possible feasible solutions can be \((x, y) = (0, 0), (1, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}), \ldots\), while for instance, \((x, y) = (-1, -1)\) is not a feasible solution.

Optimal Solution: A feasible solution that achieves the best objective value. For instance, an optimal solution for the linear program above is \((1, 0)\).

3 Geometric Interpretation

1. Variables: \((x_1, x_2, \ldots, x_n)\) correspond to a point in n-dimensional space.

2. Constraints: Each linear inequality corresponds to a plane, and the group of linear inequalities indicates the intersection of half planes, which defines a feasible region.

3. Objective: The original objective can be rewritten as

   \[ \max (\vec{c} \cdot x) = \max \sum_{i=1}^{n} (c_i x_i) \]

   The point \(\vec{c}\) represents a direction of gravity. When \(\vec{c}\) points down, the lowest point of the feasible region gives the optimal solution.

4 Canonical Form

A linear program is in canonical form if it is of the form:

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad x \geq b \\
& \quad x \geq 0
\end{align*}
\]

The form is equivalent to:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} (c_i x_i) \\
\text{for every } & \quad j = 1, 2, \ldots, m \\
& \quad \sum_{i=1}^{n} A_{ij} x_i \geq b_j \\
& \quad \text{for every } i = 1, 2, \ldots, n \\
& \quad x_i \geq 0
\end{align*}
\]
The max and min in the optimization objective can be interchanged by taking the negative of the original objective. For instance, the following linear programming problems are equivalent:

1. \( \max(2x+y), x+y \leq 1 \)
2. \( \min(-2x-y), -x-y \geq 1 \)

5 Applying Linear Programming

5.1 Fractional Knapsack

Problem statement: Given a set of \( n \) items, in which item \( i \) has a weight of \( w_i \) and a value of \( v_i \). Given a knapsack of fixed capacity \( c \) and you can put fractions of items into the knapsack. How to pack the knapsack so that it has the max value.

Variables: let \( x_i \) be the fraction of item \( i \) in the knapsack

Constraints:

1. Capacity constraint: \( \sum_{i=1}^{n} (w_i x_i) \leq c \)
2. Fraction constraint: \( 0 \leq x_i \leq 1, \forall i \in 1, 2, \ldots, n \)

Objective: \( \max \sum_{i=1}^{n} (v_i x_i) \)