Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
$L = \{ x \mid x \text{ is a positive even integer} \}$

$L$ is closed under

- addition? **yes**
- multiplication? **yes**
- subtraction? **no**
- division? **no**

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

\[
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\overline{L}_1 \\
L_1^*
\]

are regular languages.
Proof(sketck)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

Final states in $M$

care nonfinal states in $M'$

Nonfinal states in $M$

are final states in $M'$

Show $w \in L(M)$ $\Rightarrow w \in \overline{L(M)}$.

$\exists$ closed under complementation.
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 =$ $(Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 =$ $(P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' =$ $(Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = \Theta \times \Phi$

$\delta':$

$\delta'( (q_i, p_j), a) = (q_k, p_e)$ if

$\delta_1((q_i, a) = q_k) \in M_1$ and

$\delta_2((p_j, a) = p_e) \in M_2$

$F' = \{(q_i, p_j), (q_i, p_j) : q_i \in F_1 \text{ and } p_j \in F_2\}$
Example:

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Example:

1 -> a -> 2

A -> a -> B -> a

a, b

trapstate
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Regular languages are closed under

reversal \( L^R \)
difference \( L_1-L_2 \)
right quotient \( L_1/L_2 \)
homomorphism \( h(L) \)
Right quotient

Def: $L_1/L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{ a^*b^* \cup b^*a^* \}$
$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$
$L_1/L_2 = \{ x^*y^3 | x^*y^3 \text{ is odd} \}$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

If $L_i \cap L_2 \neq \emptyset$

put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$$h(bc) = 000$$

$$h(ab^*) = 1(00)^*$$
Questions about regular languages:

L is a regular language.

- Given L, \( \Sigma \), \( w \in \Sigma^* \), is \( w \in L \)?

  - Construct DFA test to see if it accepts \( w \)

- Is \( L \) empty?

  - DFS

- Is \( L \) infinite?

  - Check for cycle on path from start state to final state

- Does \( L_1 = L_2 \)?

  \[ (L_1 \cup L_2) \cup (L_1 \cap L_2) = \emptyset \]

  equivalent
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular? \( \text{yes} \)

If $L$ is infinite, is $L$ regular? \( \text{maybe} \)

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \text{aa*bb*}$
- $L_2 = \{a^n b^n | n > 0\} \text{ not}$
Prove that \( L_2 = \{ a^n b^n \mid n > 0 \} \) is not regular.

- Proof: Suppose \( L_2 \) is regular.
  \[ \Rightarrow \exists \text{ DFA } M \text{ that recognizes } L_2 \]

Consider a long string \( a^K b^K \in L_2 \) with \( K \) states, there must be a loop in the \( a's \).
Some loop in the $a$'s
say $+a$'s in the loop
$\Rightarrow$

$ak+kb$ is accepted
$\Rightarrow a$ is also accepted
$ak+kb \notin L$ !
Contradiction. DFA doesn't exist.
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

- $|xy| \leq m$
- $|y| \geq 1$
- $xy^iz \in L$ for all $i \geq 0$
To Use the Pumping Lemma to prove $L$ is not regular:

- Proof by Contradiction.
  Assume $L$ is regular.
  ⇒ $L$ satisfies the pumping lemma.
  Choose a long string $w$ in $L$,
  $|w| \geq m$.
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  The pumping lemma does not hold. Contradiction!
  ⇒ $L$ is not regular. QED.
Example \( L = \{ a^n c b^n \mid n > 0 \} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \( \Rightarrow \) the pumping lemma holds.

Choose \( w = \)
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) \}$

L is not regular.

• Proof:
  Assume L is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  - Assume $L$ is regular.
  - Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  - closure properties $\Rightarrow L'$ is regular.
  - Contradiction!
  - $L$ is not regular. QED.
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  
  Assume $L$ is regular.

  Define a homomorphism $h : \Sigma \to \Sigma^*$

  $h(a) = a \quad h(b) = a \quad h(c) = b$

  $h(L) =$
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  Assume $L$ is regular.
Example: \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.