Combining Turing Machines

We will define notation that will make it easier to look at more complicated Turing machines.

1. Given Turing Machines \( M_1 \) and \( M_2 \)
   Notation for
   - Run \( M_1 \)
   - Run \( M_2 \)

   \[
   \begin{align*}
   \text{M1} & \quad \text{M2} \\
   S & \quad S' \\
   \text{M1} \rightarrow \text{M2} & \\
   S & \quad S' \\
   \end{align*}
   \]

   \[
   z \text{ represents any symbol in } \Gamma
   \]

2. Given Turing Machines \( M_1 \) and \( M_2 \)
   Notation for
   - Run \( M_1 \)
   - If \( x \) is current symbol
     - then Run \( M_2 \)

   \[
   \begin{align*}
   \text{M1} & \quad \text{M2} \\
   S & \quad S' \\
   \text{M1} \rightarrow \text{M2} & \\
   S & \quad S' \\
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{M1} & \quad \text{M2} \\
   S & \quad S' \\
   \text{M1} \rightarrow \text{M2} & \\
   x \in \Gamma & \\
   \end{align*}
   \]

   \[
   z \text{ represents any symbol in } \Gamma
   \]

   \[
   x \text{ is an element of } \Gamma
   \]
3. Given Turing Machines M1, M2, and M3

Notation for

- Run M1
- If x is current symbol
  - then Run M2
  - else Run M3

More Notation for Simplifying Turing Machines

Suppose $\Gamma = \{a, b, c, B\}$

- z is any symbol in $\Gamma$
- x is a specific symbol from $\Gamma$

1. s - start
2. R - move right
3. L - move left

4. x - write x (and don’t move)

5. R_a - move right until you see an a

6. L_a - move left until you see an a

7. R_{\neg a} - move right until you see anything that is not an a

8. L_{\neg a} - move left until you see anything that is not an a

9. h - halt in a final state

10. \begin{align*}
a, b & \rightarrow w \\
\end{align*}

If the current symbol is a or b, let w represent the current symbol.
Example

Assume input string \( w \in \Sigma^+ \), \( \Sigma = \{a, b\} \).

If \(|w|\) is odd, then write a \( b \) at the end of the string. The tape head should finish pointing at the leftmost symbol of \( w \).

input: bab, output: babb
input: ba, output: ba

What is the running time?
**Example**
Assume input string \( w \in \Sigma^+, \Sigma = \{a, b\}, |w| > 0 \)
For each \( a \) in the string, append a \( b \) to the end of the string.
input: \( abbabb \), output: \( abbabbbb \)
The tape head should finish pointing at the leftmost symbol of \( w \).

Turing’s Thesis Any computation that can be carried out by a mechanical means can be performed by a TM.

**Definition:** An algorithm for a function \( f:D \rightarrow R \) is a TM \( M \), which given input \( d \in D \), halts with answer \( f(d) \in R \).

**Example:** \( f(x + y) = x + y \), \( x \) and \( y \) unary numbers.

\[
\begin{align*}
\text{start with:} & \quad 111+1111 \\
& \quad \uparrow \\
\text{end with:} & \quad 1111111 \\
& \quad \uparrow
\end{align*}
\]
**Example:** Copy a String, \( f(w) = w0w \), \( w \in \Sigma^* \), \( \Sigma = \{a, b, c\} \)

Denoted by \( C \)

<table>
<thead>
<tr>
<th>start with:</th>
<th>abac</th>
</tr>
</thead>
<tbody>
<tr>
<td>↑</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>end with:</th>
<th>abac0abac</th>
</tr>
</thead>
<tbody>
<tr>
<td>↑</td>
<td></td>
</tr>
</tbody>
</table>

Algorithm:

- Write a 0 at end of string
- For each symbol in string
  - make a copy of the symbol
**Example:** Shift the string that is to the left of the tape head to the right, denoted by $S_R$ (shift right).

Below, “ba” is to the left of the tape head, so shift “ba” to the right.

```
start with:      aaBbabca
         ↑       
end with:       aaBBbaca
         ↑       
```

Algorithm:

- remember symbol to the right and erase it
- for each symbol to the left do
  - shift the symbol one cell to the right
- replace first symbol erased
- move tape head to appropriate position
**Example:** Shift the string that is to the right of tape head to the left, denote by $S_L$ (shift left)

- **start with:** $babcaBba$
  - $\uparrow$
- **end with:** $bacaBBba$
  - $\uparrow$

(similar to $S_R$)

![Diagram](attachment:image.png)

```plaintext
<table>
<thead>
<tr>
<th>Start with:</th>
<th>babcaBba</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>End with:</td>
<td>bacaBBba</td>
</tr>
<tr>
<td></td>
<td>$\uparrow$</td>
</tr>
</tbody>
</table>
```
Example: Add unary numbers

This time use shift.

Example: Multiply two unary numbers, \( f(x \cdot y) = x \cdot y \), x and y unary numbers. Assume \( x, y > 0 \).

\[
\begin{align*}
\text{start with:} & \quad 1111 \cdot 11 \\
& \quad \uparrow \\
\text{end with:} & \quad 11111111 \\
& \quad \uparrow 
\end{align*}
\]