Section: Properties of Regular Languages

Example

$L = \{ a^n b a^n \mid n > 0 \}$

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\( L = \{ x \mid x \text{ is a positive even integer} \} \)

\( L \) is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[
\begin{align*}
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\overline{L}_1 \\
L_1^* 
\end{align*}
\]

are regular languages.
Proof(sket)

$L_1$ and $L_2$ are regular languages
⇒ ∃ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
⇒ closed under union

$r_1r_2$ is r.e. denoting $L_1L_2$
⇒ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$
⇒ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' =$

$\delta'$:
Example:
Regular languages are closed under

- reversal \( L^R \)
- difference \( L_1 - L_2 \)
- right quotient \( L_1 / L_2 \)
- homomorphism \( h(L) \)
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
L_1 = \{ a^*b^* \cup b^*a^* \} \\
L_2 = \{ b^n \mid n \text{ is even, } n > 0 \} \\
L_1/L_2 =
\]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$

$h(a) = 11$
$h(b) = 00$
$h(c) = 0$

$h(bc) = \quad$

$h(ab^*) = \quad$
Questions about regular languages:
L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?

• Is L empty?

• Is L infinite?

• Does L₁ = L₂?
Identifying Nonregular Languages

If a language \( L \) is finite, is \( L \) regular?

If \( L \) is infinite, is \( L \) regular?

\[
\begin{align*}
L_1 &= \{a^n b^m | n > 0, m > 0\} = \\
L_2 &= \{a^n b^n | n > 0\}
\end{align*}
\]
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \text{ for all } i \geq 0
\]
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  
  Assume $L$ is regular.
  
  \[ \Rightarrow \] $L$ satisfies the pumping lemma.
  
  Choose a long string $w$ in $L$, $|w| \geq m$.
  
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  
  The pumping lemma does not hold. Contradiction!
  
  \[ \Rightarrow \] $L$ is not regular. QED.
Example $L = \{a^n c b^n \mid n > 0\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\},$
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- Proof Outline:
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  closure properties $\Rightarrow L'$ is regular.
  Contradiction!
  $L$ is not regular. QED.
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  
  Assume $L$ is regular.

  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$

  $h(a) = a \ h(b) = a \ h(c) = b$

  $h(L) =$


Example \( L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.
Example: \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.