The Singular Value Decomposition

COMPSCI 527 — Computer Vision
Outline

1 Math Corners and the SVD: Motivation
2 Orthogonal Matrices
3 Orthogonal Projection
4 The Singular Value Decomposition
5 Principal Component Analysis
Math Corners and the SVD: Motivation

- A few math installments to get ready for later technical topics are sprinkled throughout the course
- The Singular Value Decomposition (SVD) gives the most complete \textit{geometric picture of a linear mapping}
- SVD yields orthogonal vector bases for the null space, the row space, the range, and the left null space of a matrix
- SVD leads to the \textit{pseudo-inverse}, a way to give a linear system a unique and stable approximate solution
- SVD leads to \textit{principal component analysis}, a technique to reduce the dimensionality of a set of vector data while retaining as much information as possible
- \textit{Dimensionality reduction} improves the ability of machine learning methods to generalize
Why Orthonormal Bases are Useful

- Linear space $S \subseteq \mathbb{R}^m$ (so $n \leq m$)
- $p = [p_1, \ldots, p_m]^T \in S$ (standard basis)
- $v_1, \ldots, v_n$: an orthonormal basis for $S$
  \[ v_i^T v_j = \delta_{ij} \] (Ricci delta)
- Then there exists $q = [q_1, \ldots, q_n]^T$ s.t.
  \[ p = q_1 v_1 + \ldots + q_n v_n \]
- Matrix form: $p = Vq$ where
  
  $V = [v_1, \ldots, v_n] \in \mathbb{R}^{m \times n}$

\[ \sum v_i^T p = \sum v_i^T(q_1 v_1 + q_2 v_2) = \sum q_i v_i^T v_i = q_i \]