## Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Classifying a 2-manifold** (one credit). Characterize the surface depicted in Figure II.11 in terms of genus and orientability.

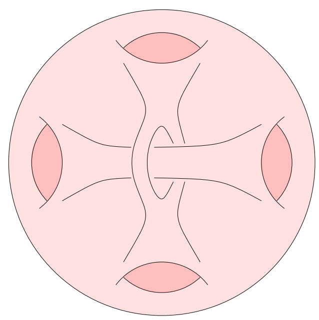

Figure II.11: A 2-manifold without boundary embedded in $\mathbb{R}^3$.

2. **Klein bottle** (two credits). Cut and paste the standard polygonal schema for the Klein bottle $(a, a, b, b)$ to obtain the polygonal schema in which opposite edges of a square are identified $(a, b, a^{-1}, b)$; see Figure II.3.
3. **Triangulation of 2-manifold** (two credits). Let $N = \{0, 1, \ldots, n-1\}$ be a set of $n$ vertices and $F \subseteq \binom{N}{3}$ a set of $m = \text{card}\, F$ triangles. Give $O(n+m)$-time algorithms for the following tasks:
   (i) decide whether or not every edge is shared by exactly two triangles;
   (ii) decide whether or not every vertex belongs to a set of triangles whose union is a disk.
4. **Intersection tests in $\mathbb{R}^3$** (two credits). Let $a, b, c \in \mathbb{R}^3$ and $u, v, w \in \mathbb{R}^3$ be the vertices of two triangles in space. Write numerical tests for the following questions:
   (i) does $u$ see $a, b, c$ form a left-turn or a right-turn?
   (ii) does the line segment with endpoints $u$ and $v$ cross the plane that passes through $a, b, c$?
   (iii) are the boundaries of the two triangles linked in $\mathbb{R}^3$?

5. **Irreducible triangulations** (three credits). An *irreducible* triangulation is one in which every edge contraction changes its topological type. Prove that the only irreducible triangulation of $\mathbb{S}^2$ is the boundary of the tetrahedron, which consists of four triangles sharing six edges and four vertices.
6. **Graphs on Möbius strip** (one credit). Is every graph that can be embedded on the Möbius strip planar?
7. **Sperner Lemma** (three credits). Let $K$ be a triangulated triangular region as in Figure II.12. We 3-color the vertices such that

   - the three corners receive three different colors;
   - the vertices on each side of the region are 2-colored.

   Prove that there is a triangle in $K$ whose vertices receive three different colors.

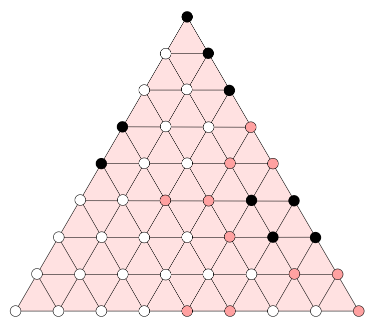

Figure II.12: Each vertex receives one of three colors, white, shaded, or black.

8. **Square distance minimization** (two credits). Let $S$ be a finite set of points in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x) = \sum_{p \in S} \|x - p\|^2$.

   (i) Show that $f$ is a quadratic function and has a unique minimum.
   (ii) At which point does $f$ attain its minimum?