Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Isomorphic homology** (one credit). Construct two topological spaces that have isomorphic homology groups but are not homotopy equivalent.
2. **Relative homology** (two credits). Let K be a triangulation of the unit square and $K_0 \subseteq K$ the subcomplex triangulating the boundary of the square. Give the ranks of the homology groups of K, K_0, and (K, K_0).
3. **Homology of the sphere** (three credits). The d-sphere can be constructed inductively by double-suspension of the $(d-1)$-sphere. In other words, if we have a triangulation of $\mathbb{S}^{d-1}$ we can get one of $\mathbb{S}^d$ by adding two new vertices and connecting each to every simplex in the triangulation.
   (i) Prove that the Euler characteristic of the d-sphere is $\chi(\mathbb{S}^d) = 1 + (-1)^d$.
   (ii) Show that for $d \geq 1$ the Betti numbers of $\mathbb{S}^d$ are $\beta_0 = \beta_d = 1$ and $\beta_i = 0$ for all $i \neq 0, d$.
   (iii) Simplify the result in (ii) using reduced homology in such a way that it also applies to the 0-sphere, $\mathbb{S}^0$.
4. **Fixed point** (two credits). Let $f : \mathbb{B}^d \to \mathbb{B}^d$ be a continuous map with the property that there is a $\delta < 1$ such that $\|f(x) - f(y)\| \leq \delta\|x - y\|$ for all points $x, y \in \mathbb{B}^d$. In words, the distance between any two points diminishes by at least a constant factor $\delta < 1$ each time we apply f. Prove that such a map f has a unique fixed point $x = f(x)$. [On orientation maps this point is usually marked as "you are here".]
5. **Klein bottle** (one credit). Show that the Betti numbers of the 2-dimensional Klein bottle are $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 1$. Which other 2-manifold has the same Betti numbers?
6. **Dunce cap** (three credits). The *dunce cap* is constructed from a piece of cloth in the shape of an equilateral triangle as follows. Orienting two edges away from a common origin we glue them to each other as prescribed by their orientation. This gives a piece of a cone with a rim (the third edge) and a seam (the glued first two edges). Now we orient the rim and glue it along the seam, again such that orientations match. The result reminds us of a snail house, perhaps.
   (i) Give a triangulation of the dunce cap.

(i) Show that the Betti numbers of the dunce cap are $\beta_0 = 1$ and $\beta_i = 0$ for all $i \neq 0$.

(ii) Show that the dunce cap is contractible but any triangulation of it is not collapsible.

7. **3-torus** (three credits). Consider the 3-*dimensional torus* obtained from the unit cube by gluing opposite faces in pairs, without twisting. That is, each point $(x, y, 0)$ is identified with $(x, y, 1)$, $(x, 0, z)$ with $(x, 1, z)$, and $(0, y, z)$ with $(1, y, z)$. Show that the Betti numbers of this space are $\beta_0 = \beta_3 = 1$ and $\beta_1 = \beta_2 = 3$.

8. **Coboundary** (one credit). Prove that the coboundary can be thought of as taking a simplex to its cofaces of one dimension higher. Formally, $\langle \varphi, \tau \rangle = 1$ iff $\langle \delta\varphi, \sigma \rangle = 1$, where $\tau < \sigma$ and $\dim \tau = \dim \sigma - 1$.