Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Hessian** (two credits). Compute the Hessian and, if defined, the index of the origin, which is critical for each function in the list below.
   - (i) $f(x_1, x_2) = x_1^2 + x_2^2$.
   - (ii) $f(x_1, x_2) = x_1 x_2$.
   - (iii) $f(x_1, x_2) = (x_1 + x_2)^2$.
   - (iv) $f(x_1, x_2, x_3) = x_1 x_2 x_3$.
   - (v) $f(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$.
   - (vi) $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2$.
2. **Approximate Morse function** (two credits). Let $\mathbb{M}$ be a geometrically perfect torus in $\mathbb{R}^3$, that is, $\mathbb{M}$ is swept out by a circle rotating about a line that lies in the same plane but does not intersect the circle. Let $f : \mathbb{M} \to \mathbb{R}$ measure height parallel to the symmetry axis and note that f is not Morse.
   - (i) Describe a Morse function $g : \mathbb{M} \to \mathbb{R}$ that differs from f by an arbitrarily small amount, $\|f - g\|_\infty < \varepsilon$.
   - (ii) Draw the Reeb graphs of both functions.
3. **Morse-Smale complex** (two credits). Let $\mathbb{M}$ be the torus in Question 2 and let $f : \mathbb{M} \to \mathbb{R}$ measure height along a direction that is almost but not quite parallel to the symmetry axis of the torus.
   - (i) Draw the Morse-Smale complex of the height function.
   - (ii) Give the chain, cycle, boundary groups defined by Floer homology.
4. **Quadrangles** (three credits). Let $\mathbb{M}$ be a 2-manifold and $f : \mathbb{M} \to \mathbb{R}$ a Morse-Smale function.
   - (i) Prove that each 2-dimensional cell of the Morse-Smale complex of f is a quadrangle. In other words, each 2-dimensional cell is an open disk whose boundary can be decomposed into four arcs each glued to an edge in the complex.
   - (ii) Draw a case in which one edge is repeated so that the disk is glued to only three edges but twice to one of the three.

5. **Distance from a point** (three credits). Let $\mathbb{M}$ be the torus swept out by a unit circle rotating at unit distance from the x_3-axis. More formally, $\mathbb{M}$ consists of all solutions to $x_1^2 + x_2^2 = (2 \pm \sqrt{1 - x_3^2})^2$ in $\mathbb{R}^3$. For a point $z \in \mathbb{R}^3$ consider the function $f_z : \mathbb{M} \to \mathbb{R}$ defined by $f_z(x) = \|x - z\|$.
   (i) Describe the set of points z for which f_z violates property (i) of a Morse function.
   (ii) Describe the set of points z for which f_z is not a Morse function.

6. **Non-simple PL critical point** (one credit). Let K be a triangulation of a 3-manifold and $f : K \to \mathbb{R}$ a generic PL function.
   (i) Assuming f is a PL Morse function, draw the lower links of the four types of simple PL critical points that can occur.
   (ii) Assuming f is not a PL Morse function, draw the lower link of a non-simple PL critical point.

7. **Lower and upper star filtrations** (one credit). Let K be a simplicial complex, $f : K \to \mathbb{R}$ a generic PL function, and $f(u_1) < f(u_2) < \ldots < f(u_n)$ the ordering of the vertices by function value. For $0 \le i \le n$ let K_i be the union of lower stars of the first i vertices and let K^i be the union of upper stars of the last $n - i$ vertices. Let $f(u_i) < t < f(u_{i+1})$.
   (i) Prove that the sublevel set for threshold t, $f^{-1}(-\infty, t]$, has the same homotopy type as K_i.
   (i) Prove that the superlevel set for threshold t, $f^{-1}[t, \infty)$, has the same homotopy type as K^i.

8. **Morse inequalities** (two credits). Recall that the unstable manifolds of a Morse function $f : \mathbb{M} \to \mathbb{R}$ are the stable manifolds of $-f$. Furthermore, if $\mathbb{M}$ is a d-manifold then an index p critical point of f is an index $d - p$ critical point of $-f$.
   (i) Use this symmetry to formulate collections of inequalities symmetric to the weak and strong Morse inequalities of f.
   (ii) Use these inequalities to prove that the Euler characteristic of $\mathbb{M}$ vanishes if d is odd.