VII.2 Shelling a 3-ball

Let K be a triangulation of a 3-ball, that is, a collection of tetrahedra sharing triangles, edges, and vertices whose union is homeomorphic to $\mathbb{B}^3$. No other, improper intersections between the tetrahedra are permitted. A *shelling* of K is an ordering of the tetrahedra such that each prefix of the ordering defines a triangulation of $\mathbb{B}^3$, and K is *shellable* if it has a shelling. It is not difficult to prove that every triangulation of $\mathbb{B}^2$ has a shelling, but the following example taken from Bing [1] shows that the same is not true for 3-balls.

The house-with-two-rooms is sketched in Figure VII.2. There are two rooms, one above the other. The only way to access the lower room is through a chimney and the only way to access the upper room is through an underground tunnel. The chimney and the tunnel are connected to the side of the house by a screen each. Now we thicken each wall, floor, ceiling and screen to one layer of bricks. All vertices belong the the boundary but edges and triangles may be on the boundary or in the interior. For a given cube we refer to the connected component of faces that belong to the boundary as *exposures*. By construction, each cube has two exposures. The union of the cubes is a 3-ball but removing any one cube destroys this property. We now decompose each cube into six

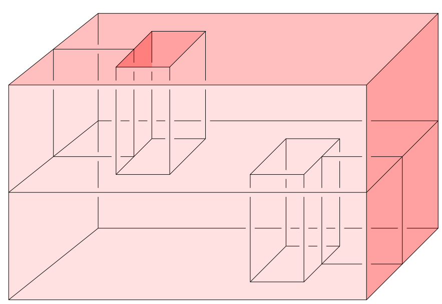

Figure VII.2: House-with-two-rooms. We can construct it from a solid block of clay without tearing or breaking.

tetrahedra in such a way that removing any one tetrahedron destroys the property of their union begin a 3-ball. In other words, no tetrahedron can be last in the shelling, which implies that the triangulation has no shelling. In order to avoid improper intersections, we first decompose each square into two triangles and then each cube into six tetrahedra in a compatible fashion. Let the *type* of a vertex be the minimum dimension of any exposure of a cube that contains the vertex. For example vertices at corners of the rooms are type 0, vertices along edges of the rooms are type 1, and the rest are type 2. Order the vertices such that type-0 vertices precede type-1 vertices which precede type-2 vertices. Now decompose each square by connecting its first vertex in the ordering to the opposite two edges. Similarly decompose each cube by connecting its first vertex in the ordering to the opposite six triangles. Again by construction each tetrahedron has two exposures, a vertex and its opposite triangle or an edge and its opposite edge. This completes the construction of the triangulation of the house-with-two-rooms that is not shellable. Since not every triangulation of $\mathbb{B}^3$ has a shelling it makes sense to ask for a decision procedure.

QUESTION. Is there a polynomial-time algorithm that decides whether or not a given triangulation of $\mathbb{B}^3$ has a shelling?

The construction of a shelling for a triangulated 2-ball is straightforward because every partial shelling is extendable and can therefore be completed [2]. This is no longer the case for the 3-ball. In other words, there are shellable triangulations of $\mathbb{B}^3$ that have non-extendable partial shellings [3]. Without a way to recognize such dead-ends we are forced into back-tracking, which takes time.

[1] R. H. BING. Some aspects of the topology of 3-manifolds related to the Poincaré conjecture. In *Lectures on Modern Mathematics II*, T. L. Saaty (ed.), Wiley, New York, 1964, 93–128.

[2] G. DANARAJ AND V. KLEE. Which spheres are shellable? In *Algorithmic Aspects of Combinatorics*, B. Alspach et al. (eds.), *Ann. Discrete Math.* **2** (1978), 33–52.

[3] G. M. ZIEGLER. Shelling polyhedral 3-balls and 4-polytopes. *Discrete Comput. Geom.* **19** (1998), 159–174.