VII.3 Geometric Realization of 2-manifolds

Recall that a geometric realization of a simplicial complex K is an embedding in which each vertex maps to a point and each (abstract) simplex maps to the (geometric) simplex spanned by the images of its vertices. The existence of a geometric realization in $\mathbb{R}^d$ can be decided using Tarski's theory of real closed fields [7]. The question is therefore decidable but Tarski's quantifier elimination method is far from practical even for small problem instances. As a special case we consider simplicial complexes K that triangulate orientable 2-manifolds and ask for geometric realizations in $\mathbb{R}^3$. Not every such K can be geometrically realized in $\mathbb{R}^3$. For example, there is a twelve-vertex triangulation of the genus-six torus that is not [1]. There is also a twelve-vertex triangulation of the genus-five torus that is not geometrically realizable even after removing one of the triangles. We can therefore take the connected sum and form arbitrarily large triangulations that have no geometric realization in $\mathbb{R}^3$ [6]. Perhaps five is the smallest genus for which this works.

QUESTION. For $1 \leq g \leq 4$, does every triangulation of the genus-g torus have a geometric realization in $\mathbb{R}^3$?

There have been attempts to prove this in the affirmative for $g = 1$ but the answer is still outstanding. The question of geometric realizability for triangulated 2-manifolds has been mentioned by Császár [3] and Grünbaum [4, Chapter 13.2]. A first serious approach to the question is described in [2]. Enumeration results can be found on Frank Lutz' web-pages [5].

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