

I.4 Planar Graphs

Although we commonly draw a graph in the plane, using tiny circles for the vertices and curves for the edges, a graph is a perfectly abstract concept. We now talk about constraints necessary to draw a graph in the plane without crossings.

Embeddings. Let $G = (V, E)$ be a simple undirected graph. A *drawing* maps every vertex $u \in V$ to a point $\varepsilon(u)$ in \mathbb{R}^2 , and it maps every edge $uv \in E$ to a path with endpoints $\varepsilon(u)$ and $\varepsilon(v)$. The drawing is an *embedding* if the points are distinct, the paths are simple and do not cross each other, and incidences are limited to endpoints. Not every graph can be drawn without crossings. The graph G is *planar* if it has an embedding in the plane. As illustrated in Figure I.13 for the complete graph of four vertices, there are many drawings of a planar graph, some with and some without crossings. Note that a graph

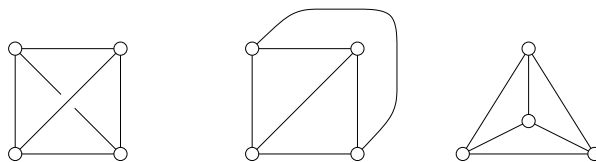


Figure I.13: Three drawings of K_4 . From left to right: a drawing that is not an embedding, and embedding with one curved edge, and a straight-line embedding.

has an embedding in the plane if and only if it has one on the 2-sphere. The latter is sometimes preferred because there is no outside region that has to be treated differently from the other regions, which are finite.

Euler's formula. A *face* of an embedding ε of G is a component in the decomposition of the plane defined by ε . We write $v = \text{card } V$, $e = \text{card } E$, and f for the number of faces. Euler's formula is a linear relation between the three numbers.

EULER RELATION. Every embedding of a connected graph in the plane satisfies $v - e + f = 2$.

PROOF. Choose a spanning tree (V, T) of (V, E) . It has v vertices, $\text{card } T = v - 1$ edges, and one face. We have $v - (v - 1) + 1 = 2$, which proves the formula

if G is a tree. Otherwise, draw the remaining edges, one at a time. Each edge decomposes one face into two. The number of vertices does not change, e increases by one, and f increases by one. Since the graph satisfies the claimed relation before drawing the edge it satisfies the relation also after drawing the edge. \square

If the graph consists of $c \geq 1$ connected components we have $v - e + f = c + 1$. Note that the Euler Relation implies that the number of faces is the same for all embeddings and is therefore a property of the graph. We get bounds on the number of edges and faces, in terms of the number of vertices, by considering *maximally connected* graphs for which adding any one edge would violate planarity. Every face of a maximally connected planar graph with three or more vertices is necessarily a triangle, for if there is a face with more than three edges we can add a path that crosses none of the earlier paths. Let $v \geq 3$ be the number of vertices, as before. Since every face has three edges and every edge belong to two triangles, we have $3f = 2e$. We use this relation to rewrite the Euler Relation: $v - e + \frac{2e}{3} = 2$ and $v - \frac{3f}{2} + f = 2$ and hence $e = 3v - 6$ and $f = 2v - 4$. Every planar graph can be completed to a maximally connected planar graph, which implies that it has at most these numbers of edges and faces.

Note that the sum of vertex degrees is twice the number of edges, and therefore $\sum_u \deg u < 6n$. It follows that every planar graph has a vertex of degree less than six. This observation suggests inductive approaches to various questions about planar graphs, such as coloring the vertices and constructing straight-line embeddings.

Non-planarity. We can use the Euler Relation to prove that the complete graph of five vertices and the complete bipartite graph of three plus three vertices are not planar. Consider first K_5 , which is drawn in Figure I.14, left. It has $v = 5$ vertices and $e = 10$ edges, contradicting the upper bound of

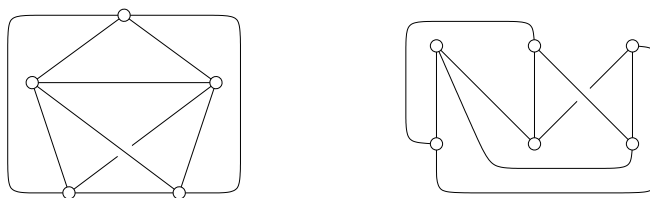


Figure I.14: K_5 on the left and $K_{3,3}$ on the right, each drawn with the unavoidable one crossing.

at most $3v - 6 = 9$ edges for maximally connected planar graphs. Consider second $K_{3,3}$, which is drawn in Figure I.14, right. It has $v = 6$ vertices and $e = 9$ edges. Each cycle has even length, which implies that each face has four or more edges. We get $4f \leq 2e$ and $e \leq 2v - 4 = 8$ after plugging the inequality into the Euler Relation, again a contradiction.

In a sense, K_5 and $K_{3,3}$ are the quintessential non-planar graphs. Two graphs are *homeomorphic* if one can be obtained from the other by a sequence of operations, each deleting a degree-2 vertex and merging their two edges into one or doing the inverse.

KURATOWSKI THEOREM. A simple graph is planar iff no subgraph is homeomorphic to K_5 or to $K_{3,3}$.

The proof of this result is omitted. The remainder of this section focuses on straight-line embeddings of planar graphs.

Convex combinations. Two points $a_0 \neq a_1$ define a unique line that passes through both. Each point on this line can be written as $x = (1 - \lambda)a_0 + \lambda a_1$, for some $\lambda \in \mathbb{R}$. For $\lambda = 0$ we have $x = a_0$ and for $\lambda = 1$ we have $x = a_1$. The point belongs to the line segment connecting a_0 to a_1 iff $0 \leq \lambda \leq 1$. If we have more than two points we repeat the construction by adding all points $y = (1 - \lambda)x + \lambda a_2$ for which $0 \leq \lambda \leq 1$, and so on, as illustrated in Figure I.15. Given $k + 1$ points a_0, a_1, \dots, a_k , we can do the same construction in one step,

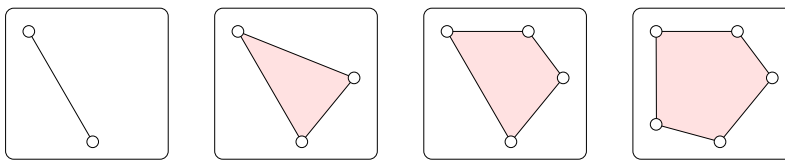


Figure I.15: From left to right: the construction of the convex hull of five points by adding one point at a time.

calling a point $x = \sum_{i=0}^k \lambda_i a_i$ a *convex combination* of the a_i if $\lambda_i \geq 0$ for all $0 \leq i \leq k$ and $\sum_{i=0}^k \lambda_i = 1$. The set of convex combinations is the *convex hull* of the a_i .

Let now K be a triangulation of a disk. In other words, the edge-skeleton consisting of the vertices and edges of K is a planar graph such that each interior face is bounded by three edges and their union is homeomorphic to

\mathbb{B}^2 . Letting V be the set of vertices, we call $f : V \rightarrow \mathbb{R}$ a (strictly) *convex combination function* if for each interior vertex $u \in V$ there are real numbers $\lambda_{uv} > 0$ satisfying

$$\sum_v \lambda_{uv} = 1; \tag{I.1}$$

$$\sum_v \lambda_{uv} f(v) = f(u), \tag{I.2}$$

where both sums are over all neighbors v of u . Similarly, we call $\varepsilon : V \rightarrow \mathbb{R}^2$ a (strictly) *convex combination mapping* if there are strictly positive real numbers λ_{uv} such that (I.1) and (I.2) hold for ε .

Straight-line embedding. It is not difficult to show that every straight-line embedding of the edge-skeleton of K defines a convex combination mapping. We will show that the reverse is also true provided the boundary vertices form a strictly convex polygon.

TUTTE'S THEOREM. If $\varepsilon : V \rightarrow \mathbb{R}^2$ is a convex combination mapping that maps the boundary vertices to a strictly convex polygon then drawing the straight edges between the images of the vertices is a straight-line embedding of the edge-skeleton of K .

We will give the proof in three steps, which we now prepare with a few observations. A non-zero vector $p \in \mathbb{R}^2$ and a real number c define a function $f(x) = \langle x, p \rangle + c$, which is positive on one side of the line $f^{-1}(0)$ and negative on the other. Suppose ε is a convex combination mapping. Then there are positive real numbers λ_{uv} such that $\varepsilon(u) = \sum_v \lambda_{uv} \varepsilon(v)$ for each interior vertex u . We therefore have

$$\begin{aligned} f(\varepsilon(u)) &= \left\langle \sum_v \lambda_{uv} \varepsilon(v), p \right\rangle + \sum_v \lambda_{uv} c \\ &= \sum_v \lambda_{uv} f(\varepsilon(v)). \end{aligned}$$

In words, f is a convex combination function.

A *separating edge* of K is an interior edge that connects two boundary vertices. It is convenient to assume there are no separating edges, and if there is one we can split K into two and do the argument for each piece. Call a path *interior* if all its points are interior except possibly its two endpoints. Under the assumption of no separating edges, every interior vertex u can be connected

to every boundary vertex by an interior path. Indeed, since K is finite, we can find an interior path that connects u to a first boundary vertex w . Let w_0 and w_1 be the neighboring boundary vertices. Since none of the edges separate, the neighbors of w form a unique interior path connecting w_0 to w_1 . It follows that there is an interior path connecting u to w_0 . By repeating the argument substituting w_0 for w we eventually see that u has an interior path to every boundary vertex. This can be used to prove the following result.

MAXIMUM PRINCIPLE. Let K be a triangulation of \mathbb{B}^2 without separating edges and $f : V \rightarrow \mathbb{R}$ a convex combination function. If $f(u) \geq f(w)$ for an interior vertex u and every boundary vertex w then $f(u) = f(v)$ for every vertex $v \in V$.

To see this, let u_0 be the interior vertex that maximizes f . By assumption, $f(u_0) \geq f(v)$ for all $v \in V$. Since f is a convex combination function, all neighbors of u_0 have the same function value. Repeating this argument for the neighbors of u_0 and their neighbors and so on implies the Maximum Principle.

Proof of Tutte's Theorem. We now present the proof in three steps. First, all interior vertices u of V map to the interior of the strictly convex polygon whose corners are the images of the boundary vertices. To see this, choose $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$ such that the line $f^{-1}(0)$ defined by $f(x) = \langle x, p \rangle + c$ passes through a boundary edge and $f(\varepsilon(w)) > 0$ for all boundary vertices other than the endpoints of that edge. Then $f(\varepsilon(u)) > 0$ else the Maximum Principle would imply $f(\varepsilon(v)) = 0$ for all vertices. Repeating this argument for all edges of the convex polygon implies that all interior vertices u have $\varepsilon(u)$ in the interior of the polygon. This implies in particular that each triangle incident to a boundary edge is non-degenerate, that is, its three vertices are not collinear.

Second, letting yuv and zuv be the two triangles sharing the interior edge uv in K , the points $\varepsilon(y)$ and $\varepsilon(z)$ lie on opposite sides of the line $f^{-1}(0)$ that passes through $\varepsilon(u)$ and $\varepsilon(v)$. To see this assume $f(\varepsilon(y)) > 0$ and find a strictly rising path connecting y to the boundary. It exists because $f(\varepsilon(y)) > f(\varepsilon(u))$ so one of the neighbors of y has strictly larger function value, and the same is true for the next vertex on the path and so on. Similarly, find a strictly falling path connecting u to the boundary and the same for v , as illustrated in Figure I.16. The rising path does not cross the falling paths, but the two falling paths may share a vertex, as in Figure I.16. In either case, we get a piece of the triangulation bounded by vertices with non-positive function values. Other than u and v all other vertices in this boundary have strictly negative function

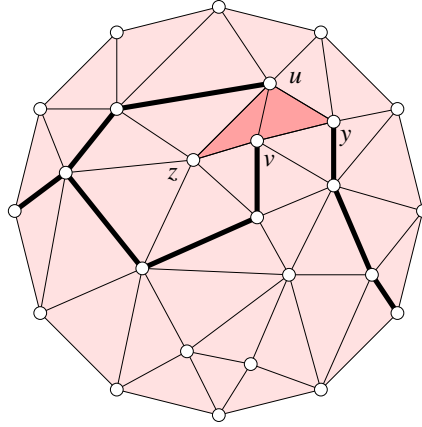


Figure I.16: One strictly rising and two strictly falling paths connecting y , u , and v to the boundary.

values. If z belongs to the boundary of this piece it has strictly negative function value simply because it differs from u and v . Else it belongs to the interior of the piece and we have $f(\varepsilon(z)) < 0$ by the Maximum Principle. We note that this argument uses $f(\varepsilon(y)) > 0$ in an essential manner. To show that this assumption is justified, we connect yuv by a sequence of triangles to one incident to a boundary edge. In this sequence, any two contiguous triangles share an edge. As observed in Step 1, the image of the last triangle is non-degenerate. Going backward this implies that the image of the second to the last triangle is non-degenerate and so on. Finally, the image of yuv is non-degenerate, as required.

Third, the images of any two triangles in K have disjoint interiors. To get a contradiction assume x is a point in the common interiors of two such images, σ and τ . Choose a half-line that emanates from x and avoids all images of vertices. It defines a sequence of triangles starting with σ and ending at the triangle v incident to the boundary edge whose image crosses the half-line. Similarly, the half-line defines another sequence of triangles starting with τ and ending with the same triangle v . Going back from v we pass from one quadrangle to the next. Each step is unambiguous which implies $\sigma = \tau$.

Bibliographic notes. Graphs that can be drawn in the plane without crossings arise in a number of applications, including geometric modeling, geographic information systems, and others. We refer to [4] for a collection of mathemat-

ical and algorithmic results specific to planar graphs. The fact that planar graphs have straight-line embeddings has been known long before Tutte's Theorem. Early last century, Steinitz showed that every 3-connected planar graph is the edge-skeleton of a convex polytope in \mathbb{R}^3 [5]. This skeleton can be projected to \mathbb{R}^2 to give a straight-line embedding. In the 1930s, Koebe proved that every planar graph is the intersection graph of a collection of possibly touching but not otherwise overlapping closed disks in \mathbb{R}^2 [3]. We get a straight-line embedding by connecting the centers of touching disks. This result has been rediscovered by Andreev and independently by Thurston and is known under all three names. Probably the simplest proof that every planar graph has a straight-line embedding uses an inductive argument adding a vertex of degree at most five at each step [1]. This leads to huge differences in lengths and turns out to be less useful in practice for this reason. The original theorem by Tutte is for coefficients λ_{uv} equal to one over the degree of u [6]. The more general version and the proof presented in this section are taken from the more recent paper by Floater [2]. The theorem can be turned into a linear system with equally many equations as unknowns. By the Euler Relation this system is sparse and thus permits efficient solutions.

- [1] I. FÁRY. On straight line representation of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1948), 229–233.
- [2] M. S. FLOATER. One-to-one piecewise linear mappings over triangulations. *Math. Comput.* **72** (2003), 685–696.
- [3] P. KOEBE. Kontaktprobleme der konformen Abbildung. *Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl.* **88** (1936), 141–164.
- [4] T. NISHIZEKI AND N. CHIBA. *Planar Graphs: Theory and Algorithms*. North-Holland, Amsterdam, the Netherlands, 1988.
- [5] E. STEINITZ. *Polyeder und Raumeinteilung*. In *Enzykl. Math. Wiss., Part 3AB12* (1922), 1–139.
- [6] W. T. TUTTE. How to draw a graph. *Proc. London Math. Soc.* **13** (1963), 743–768.