II.1 Two-dimensional Manifolds

The term 'surface' is technically less specific but mostly used synonymous to '2-manifold' for which we will give a concrete definition.

Topological 2-manifolds. Consider the open disk of points at distance less than one from the origin, $D = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$. It is homeomorphic to $\mathbb{R}^2$, as for example established by the homeomorphism $f : D \to \mathbb{R}^2$ defined by $f(x) = x/(1 - \|x\|)$. Indeed, every open disk is homeomorphic to the plane.

Definition. A 2-*manifold (without boundary)* is a topological space $\mathbb{M}$ whose points all have open disks as neighborhoods. It is *compact* if every open cover has a finite subcover.

Intuitively, this means that $\mathbb{M}$ looks locally like the plane everywhere. Examples of non-compact 2-manifolds are $\mathbb{R}^2$ itself and open subsets of $\mathbb{R}^2$. Examples of compact 2-manifolds are shown in Figure II.1, top row. We get 2-*manifolds*

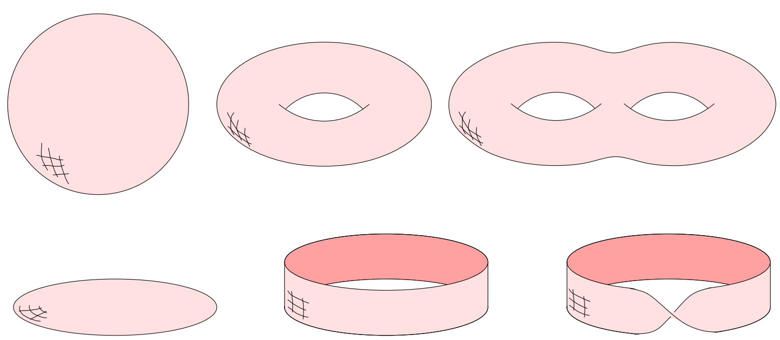

Figure II.1: Top from left to right: the sphere, $\mathbb{S}^2$, the torus, $\mathbb{T}^2$, the double torus, $\mathbb{T}^2 \# \mathbb{T}^2$. Bottom from left to right: the disk, the cylinder, the Möbius strip.

with boundary by removing open disks from 2-manifolds with boundary. Alternatively, we could require that each point has a neighborhood homeomorphic to either D or to half of D obtained by removing all points with negative first coordinate. The *boundary* of a 2-manifold with boundary consists of all points x of the latter type. Within the boundary, the neighborhood of every point x is an open interval, which is the defining property of a 1-*manifold*. There is only one type of compact 1-manifold, namely the circle. If $\mathbb{M}$ is compact, this implies that its boundary is a collection of circles. Examples of 2-manifolds with boundary are the (closed) disk, the cylinder, and the Möbius strip, all illustrated in Figure II.1, bottom row.

We get new 2-manifolds from old ones by gluing them to each other. Specifically, remove an open disk each from two 2-manifolds, $\mathbb{M}$ and $\mathbb{N}$, find a homeomorphism between the two boundary circles, and identify corresponding points. The result is the *connected sum* of the two manifolds, denoted as $\mathbb{M}\#\mathbb{N}$. Forming the connected sum with the sphere does not change the manifold since it just means replacing one disk by another. Adding the torus is the same as attaching the cylinder at both boundary circles after removing two open disks. Since this is like adding a handle we will sometimes refer to the torus as the sphere with one handle, the double torus as the sphere with two handles, etc.

Orientability. Of the examples we have seen so far, the Möbius strip has the curious property that it seems to have two sides locally at every interior point but there is only one side globally. To express this property intrinsically, without reference to the embedding in $\mathbb{R}^3$, we consider a small, oriented circle inside the strip. We move it around without altering its orientation, like a clock whose fingers keep turning in the same direction. However, if we slide the clock once around the strip its orientation is the reverse of what it used to be and we call the path of its center an *orientation-reversing* closed curve. There are also

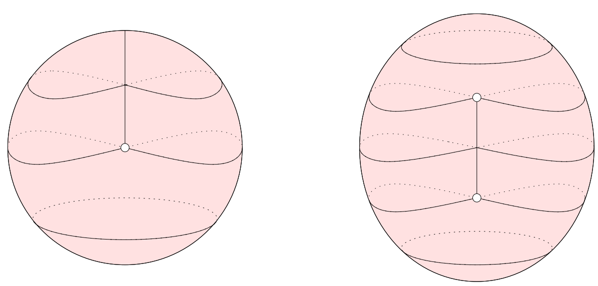

Figure II.2: Left: the projective plane, $\mathbb{P}^2$, obtained by gluing a disk to a Möbius strip. Right: the Klein bottle, $\mathbb{K}^2$, obtained by gluing two Möbius strips together. The vertical lines are self-intersections that are forced by placing the 2-manifolds in $\mathbb{R}^3$. They are topologically not important.

orientation-preserving closed curves in the Möbius strip, such as the one that goes around the strip twice. If all closed curves in a 2-manifold are orientation-preserving then the 2-manifold is *orientable*, else it is *non-orientable*.

Note that the boundary of the Möbius strip is a single circle. We can therefore glue the strip to a sphere or a torus after removing an open disk from the latter. This operation is often referred to as adding a *cross-cap*. In the first case we get the *projective plane*, the sphere with one cross-cap, and in the second case we get the *Klein bottle*, the sphere with two cross-caps. Both cannot be embedded in $\mathbb{R}^3$, so we have to draw them with self-intersections, but these should be ignored when we think about these surfaces.

Classification. As it turns out, we have seen examples of each major kind of compact 2-manifold. They have been completely classified about a century ago by cutting and gluing to arrive at a unique representation for each type. This representation is a convex polygon whose edges are glued in pairs, called a *polygonal schema*. Figure II.3 shows that the sphere, the torus, the projective plane, and the Klein bottle can all be constructed from the square. More

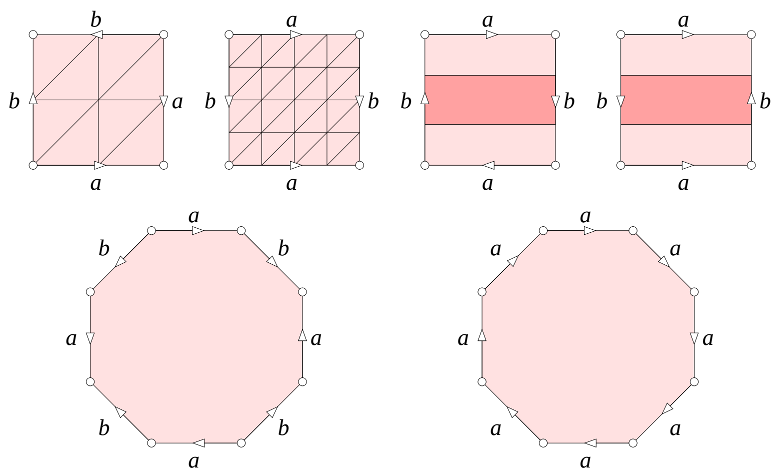

Figure II.3: Top from left to right: the sphere, the torus, the projective plane, and the Klein bottle. After removing the (darker) Möbius strip from the last two, we are left with a disk in the case of the projective plane and another Möbius strip in the case of the Klein bottle. Bottom: the polygonal schema in standard form for the double torus on the left and the double Klein bottle on the right.

generally, we have a $4g$-gon for a sphere with g handles and a $2g$-gon for a sphere with g cross-caps attached to it. The gluing pattern is shown in the second row of Figure II.3. Note that the square of the torus is in standard form but that of the Klein bottle is not.

Classification Theorem. The two infinite families $\mathbb{S}^2, \mathbb{T}^2, \mathbb{T}^2\#\mathbb{T}^2, \ldots$ and $\mathbb{P}^2, \mathbb{P}^2, \mathbb{P}^2\#\mathbb{P}^2, \ldots$ exhaust the family of compact 2-manifolds without boundary.

To get a classification of compact 2-manifolds with boundary we can take one without boundary and make h holes by removing the same number of open disks. Each starting 2-manifold and each $h \geq 1$ give a different surface and they exhaust all possibilities.

Triangulations. To triangulate a 2-manifold we decompose it into triangular regions, each a disk whose boundary circle is cut at three points into three paths. We may think of the region and its boundary as the homeomorphic image of a triangle. By taking a geometric triangle for each region and arranging them so they share vertices and edges the same way as the regions we obtain a piecewise linear model which is a *triangulation* if it is homeomorphic to the 2-manifold. See Figure II.4 for a triangulation of the sphere. The condition

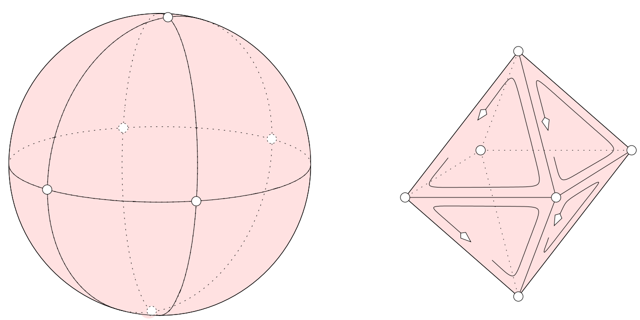

Figure II.4: The sphere is homeomorphic to the surface of an octahedron, which is a triangulation of the sphere.

of homeomorphism requires that any two triangles are either disjoint, share an edge, or share a vertex. Sharing two edges is not permitted for then the two triangles would be the same. It is also not permitted that two vertices of a triangle are the same. To illustrate these conditions we note that the triangulation of the first square in Figure II.3 is not a valid triangulation of the sphere, but the triangulation of the second square is a valid triangulation of the torus.

Given a triangulation of a 2-manifold $\mathbb{M}$, we may orient each triangle. Two triangles sharing an edge are *consistently oriented* if they induce oppose orientations on the shared edge, as in Figure II.4. Then $\mathbb{M}$ is orientable iff the triangles can be oriented in such a way that every adjacent pair is consistently oriented.

Euler characteristic. Recall that a triangulation is a collection of triangles, edges, and vertices. We are only interested in finite triangulations. Letting v, e, and f be the numbers of vertices, edges, and triangles, the *Euler characteristic* is their alternating sum, $\chi = v - e + f$. We have seen that the Euler characteristic of the sphere is $\chi = 2$, no matter how we triangulate. More generally, the Euler characteristic is independent of the triangulation for every 2-manifold.

Euler Characteristic of Compact 2-manifolds. A sphere with g handles has $\chi = 2 - 2g$ and a sphere with g cross-caps has $\chi = 2 - g$.

The number g is the *genus* of $\mathbb{M}$; it is the maximum number of disjoint closed curves along which we can cut without disconnecting $\mathbb{M}$. To see this result we may triangulate the polygonal schema of $\mathbb{M}$. For a sphere with g handles we have $f = 1$ region, $e = 2g$ edges, and $v = 1$ vertex. Further decomposing the edges and regions does not change the alternating sum, so we have $\chi = 2 - 2g$. For a sphere with g cross-caps we have $f = 1$ region, $e = g$ edges, and $v = 1$ vertex giving $\chi = 2 - g$. This result suggests an easy algorithm to recognize a compact 2-manifold given by its triangulation. First search all triangles and orient them consistently as you go until you either succeed, establishing orientability, or you encounter a contradiction, establishing non-orientability. Thereafter count the vertices, edges, and triangles, and the alternating sum uniquely identifies the 2-manifold if there is no boundary. Else count the holes, this time by searching the edges that belong to only one triangle each. For each additional hole the Euler characteristic decreases by one, giving $\chi = 2 - 2g - h$ in the orientable case and $\chi = 2 - g - h$ in the non-orientable case. The genus, g, and the number of holes, h, identify a unique 2-manifold with boundary within the orientable and the non-orientable classes.

Doubling. The compact, non-orientable 2-manifolds can be obtained from the orientable 2-manifolds by identifying points in pairs. We go the other

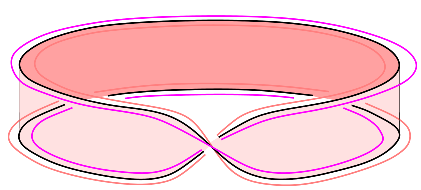

Figure II.5: Doubling a Möbius strip produces a cylinder.

direction, constructing orientable from non-orientable manifolds; see Figure

II.5. Imagine a triangulation of a non-orientable 2-manifold $\mathbb{N}$ in $\mathbb{R}^3$, drawn with possible self-intersections, which we ignore. Make two copies of each triangle, edge, and vertex off-setting them slightly, one on either side of the manifold. Here sidedness is local and therefore well defined. The triangles fit together locally, and because $\mathbb{N}$ is orientable, they fit together to form the triangulation of a connected 2-manifold, $\mathbb{M}$. It is orientable because one side is consistently facing $\mathbb{N}$. Since all triangles, edges, vertices are doubled we have $\chi(\mathbb{M}) = 2\chi(\mathbb{N})$. Using the relation between genus and Euler characteristic we have $\chi(\mathbb{N}) = 2 - g(\mathbb{N})$ and therefore $\chi(\mathbb{M}) = 4 - 2g(\mathbb{N}) = 2 - 2g(\mathbb{M})$. It follows that $\mathbb{M}$ has $g(\mathbb{M}) = g(\mathbb{N}) - 1$ handles. Hence, the doubling operation constructs the sphere from the projective plane, the torus from the Klein bottle, etc.

Bibliographic notes. The confusing aspects of non-orientable 2-manifolds have been captured in a delightful novel about the life within such a surface [1]. The classification of compact 2-manifolds is sometimes credited to Brahana [2] and other times to Dehn and Heegard [3]. The classification of 3-manifolds, on the other hand, is an ongoing project within mathematics. With the proof of the Poincaré conjecture by Perelman, there is new hope that this can be soon accomplished. In contrast, recognizing whether two triangulated 4-manifolds are homeomorphic is undecidable [4]. The classification of manifolds beyond dimension three is therefore hopeless.

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