II.2 Fundamental Group

In this section we talk about paths and loops, trying to get around on surfaces and elsewhere without getting confused. We use this topic as an excuse to introduce the first substantially algebraic tool used in topology.

Paths and products. Recall that a path in a topological space is a continuous map from the unit interval. Call two paths $p, q : [0,1] \to \mathbb{X}$ from $x = p(0) = q(0)$ to $y = p(1) = q(1)$ *equivalent* if there is a continuous map $H : [0,1] \times I \to \mathbb{X}$, where $I = [0,1]$, such that $H(s,0) = p(s)$ and $H(s,1) = q(s)$ for all $s \in [0,1]$ as well as $H(0,t) = x$ and $H(1,t) = y$ for all $t \in I$. We write $p \sim q$ if p and q are equivalent and think of H as a continuous deformation of p to q that keeps the endpoints fixed. The deformation happens within $\mathbb{X}$ but is not hindered by self-intersections. Recall that an equivalence relation partitions. In this case, it partitions the set of paths from x to y. Given such a path p, we denote the class of paths equivalent to p by $[p]$. The *product* of two paths p and q is defined if $p(1) = q(0)$, namely

$$pq(s) = \begin{cases} p(2s) & \text{for } 0 \le s \le \frac{1}{2}, \\ q(2s-1) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

We are interested in equivalence classes more than in individual paths. Fortunately, the product operation is compatible with the concept of equivalence, $p_0 \sim q_0$ and $p_1 \sim q_1$ implies $p_0 p_1 \sim q_0 q_1$. In other words, $[p][q] = [pq]$ is well defined because it does not depend on the representatives p and q of the two classes. Defining the product for classes of paths, we check the group axioms for this operation.

Associativity. Note that $p(qr) \neq (pq)r$, even if all products are defined, but we can reparametrize to make them equal so they are certainly equivalent. Hence $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, where $\alpha = [p]$, $\beta = [q]$, and $\gamma = [r]$.

Neutral element. Write $1_x(s) = x$ for the constant path at $x \in \mathbb{X}$ and let $\varepsilon_x = [1_x]$. Clearly $\varepsilon_x \alpha = \alpha$ and $\alpha \varepsilon_y = \alpha$ assuming $\alpha = [p]$ with $x = p(0)$ and $y = p(1)$.

Inverse element. Let $p^{-1}(s) = p(1-s)$ be the same path but going backwards and define $\alpha^{-1} = [p^{-1}]$. Then every path in $\alpha\alpha^{-1}$ is equivalent to 1_x and every path in $\alpha^{-1}\alpha$ is equivalent to 1_y.

In summary, we almost have a group but the product operation is not always defined.

Loops. As a remedy we consider paths that start and end at the same point. A *loop* is a path $p : [0,1] \to \mathbb{X}$ with $x = p(0) = p(1)$; it is said to be *based* at x. Now the product operation is always defined and $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \varepsilon_x$.

DEFINITION. The *fundamental group*, denoted as $\pi(\mathbb{X}, x)$, consists of all equivalence classes of loops based at x and the product operation between them.

In most cases, the point x is not important. Indeed suppose points $x, y \in \mathbb{X}$ are connected by a path m. Then we can map each loop p based at x to the loop $m^{-1}pm$ based at y. In the other direction we map the path q based at y to mqm^{-1} based at x; see Figure II.6. Extending the construction to equivalence

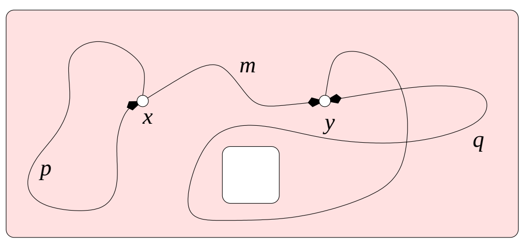

Figure II.6: The loops p and mqm^{-1} are based at x and the loops q and $m^{-1}pm$ are based at y. Since q goes around the hole and p does not, the two are not equivalent.

classes we get $u : \pi(\mathbb{X}, x) \to \pi(\mathbb{X}, y)$ defined by $u(\alpha) = \mu^{-1}\alpha\mu$, where $\mu = [m]$ and $\alpha = [p]$. It is easy to see that u commutes with the product operation, that is, $u(\alpha)u(\beta) = u(\alpha\beta)$. In words, u is a homomorphism. Symmetrically, we get a homomorphism $v : \pi(\mathbb{X}, y) \to \pi(\mathbb{X}, x)$ defined by $v(\gamma) = \mu\gamma\mu^{-1}$. The composition $v \circ u$ maps α to $\mu(\mu^{-1}\alpha\mu)\mu^{-1} = \alpha$ and is therefore the identity on $\pi(\mathbb{X}, x)$. Symmetrically, $u \circ v$ is the identity on $\pi(\mathbb{X}, y)$. Hence u and v are isomorphisms. To summarize, if $\mathbb{X}$ is path-connected then $\pi(\mathbb{X}, x)$ and $\pi(\mathbb{X}, y)$ are isomorphic for any two points $x, y \in \mathbb{X}$. It thus makes sense to leave out the base point and write $\pi(\mathbb{X})$, but we need to keep in mind that there is no canonical isomorphism between the fundamental groups since it depends on the connecting path.

Induced homomorphisms. An important idea in algebraic topology is that continuous maps between spaces induce homomorphisms between groups. Let $f : \mathbb{X} \to \mathbb{Y}$ be such a map. As illustrated in Figure II.7, any two equivalent paths $p, q : [0,1] \to \mathbb{X}$ map to equivalent paths $f(p), f(q) : [0,1] \to \mathbb{Y}$.

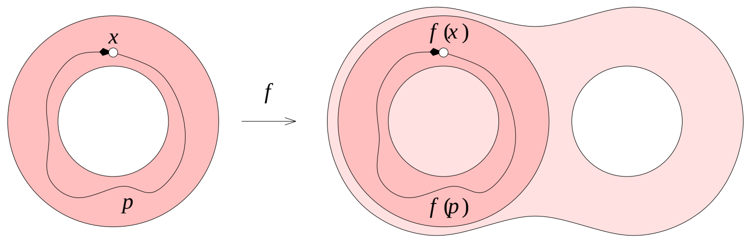

Figure II.7: The annulus on the left is injectively mapped into the annulus on the right. Since this operation fills the hole, all non-trivial loops on the left map to $\varepsilon_{f(x)}$ on the right.

Hence f induces the homomorphism $f_* : \pi(\mathbb{X}, x) \to \pi(\mathbb{X}, f(x))$ which maps the class $\alpha = [p]$ to the class $f_*(\alpha) = [f(p)]$, which contains all paths $f(p)$ with $p \in \alpha$ but possibly more. In Figure II.7, each class in $\pi(\mathbb{X}, x)$ is mapped to the neutral element in $\pi(\mathbb{X}, f(x))$. Even though both groups are isomorphic, this particular homomorphism is not an isomorphism. It is clear that if f is a homeomorphism then f_* is an isomorphism. On the other hand, the example shows that f being injective does not imply that f_* is a monomorphism (injective homomorphism). Similarly, f being surjective does not imply that f_* is an epimorphism (surjective homomorphism).

Retracts. The fundamental group of the annulus is infinitely cyclic, consisting of all classes α^k, k and integer, where α goes around the annulus once. In contrast, the fundamental group of the disk is trivial, consisting only of one element. We can use this difference to prove Brouwer's Theorem in the plane.

Call a subset $\mathbb{A} \subseteq \mathbb{X}$ a *retract* of $\mathbb{X}$ if there is a continuous map $r : \mathbb{X} \to \mathbb{A}$ with $r(a) = a$ for all $a \in \mathbb{A}$. The map r is called a *retraction.* A simple example is the torus and a closed curve in the torus. Letting $i : \mathbb{A} \to \mathbb{X}$ be the inclusion map, $i(a) = a$ for all $a \in \mathbb{A}$, we have two induced homomorphisms for each point $a \in \mathbb{A}$,

$$\begin{aligned} i_* &: \pi(\mathbb{A}, a) \to \pi(\mathbb{X}, a), \\ r_* &: \pi(\mathbb{X}, a) \to \pi(\mathbb{A}, a). \end{aligned}$$

Since $r \circ i : \mathbb{A} \to \mathbb{A}$ is the identity, $r_* \circ i_* : \pi(\mathbb{A}, a) \to \pi(\mathbb{A}, a)$ is the identity homomorphism. This implies that i_* is a monomorphism and r_* is an epimorphism.

Brouwer's Theorem. A continuous map $f : \mathbb{B}^2 \to \mathbb{B}^2$ has at least one fixed point $x = f(x)$.

Proof. To get a contradiction assume $x \neq f(x)$ for all $x \in \mathbb{B}^2$. Consider the half-line of points $x + \lambda(x - f(x))$, for all $\lambda \geq 0$, and let $r(x)$ be the point at which this half-line intersects the boundary circle; see Figure II.8. The map r is

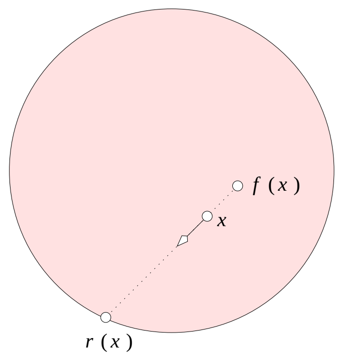

Figure II.8: Mapping a point x of the disk to a point $r(x)$ on the circle bounding the disk. Note that $r(x) = x$ if x belongs to the circle.

continuous because f is continuous and without fixed point. We have $r(a) = a$ for every boundary point $a \in \mathbb{S}^1$, hence $r : \mathbb{B}^2 \to \mathbb{S}^1$ is a retraction. But then r_* is an epimorphism, which is impossible because $\pi(\mathbb{B}^2)$ contains one element while $\pi(\mathbb{S}^1)$ contains infinitely many. ◻

Homotopy. A retraction does not preserve the fundamental group but there is a stronger version that does.

Definition. Two continuous maps $f, g : \mathbb{X} \to \mathbb{Y}$ are *homotopic* if there is a continuous map $H : \mathbb{X} \times I \to \mathbb{Y}$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in \mathbb{X}$.

We call H a *homotopy*. Noting that this defines an equivalence relation we write $f \simeq g$ if they are homotopic and refer to the equivalence classes as *homotopy classes* of maps. We can think of I as a time-interval and the homotopy as a time-series of functions $f_t : \mathbb{X} \to \mathbb{Y}$ defined by $f_t(x) = H(x, t)$.

A subset $\mathbb{A} \subseteq \mathbb{X}$ is a *deformation retract* of $\mathbb{X}$ if there exists a retraction $r : \mathbb{X} \to \mathbb{A}$ that is homotopic to the identity map on $\mathbb{X}$. Here the map r is called a *deformation retraction*. Letting $i : \mathbb{A} \to \mathbb{X}$ be the inclusion map, we have seen before that $r_* \circ i_* : \pi(\mathbb{A}, a) \to \pi(\mathbb{A}, a)$ is the identity homomorphism. By assumption, the reverse composition $i \circ r : \mathbb{X} \to \mathbb{X}$ is homotopic to the identity, which implies that $i_* \circ r_* : \pi(\mathbb{X}, a) \to \pi(\mathbb{X}, a)$ is again the identity homomorphism. It follows that i_* and r_* are isomorphisms. In other words, the fundamental group of the deformation retract $\mathbb{A}$ is isomorphic to the fundamental group of $\mathbb{X}$. We will use this fact repeatedly in this book. An insubstantial generalization of the notion of deformation retract is the following.

Definition. Two topological spaces $\mathbb{X}$ and $\mathbb{Y}$ are *homotopy equivalent* if there are continuous maps $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$ such that $g \circ f$ is homotopic to the identity on $\mathbb{X}$ and $f \circ g$ is homotopic to the identity on $\mathbb{Y}$.

If $\mathbb{X}$ and $\mathbb{Y}$ are homotopy equivalent we write $\mathbb{X} \simeq \mathbb{Y}$ and say they have the same *homotopy type*. For example if $r : \mathbb{X} \to \mathbb{A}$ is a deformation retraction then $f = r$ and $g = i$ are continuous maps that satisfy the conditions and thus establish $\mathbb{X} \simeq \mathbb{A}$.

A path-connected topological space $\mathbb{X}$ is *simply connected* if $\pi(\mathbb{X})$ is trivial. Furthermore, $\mathbb{X}$ is *contractible* if it has the homotopy type of a point. Clearly every contractible space is simply connected but the reverse is not true. For example, $\mathbb{S}^2$ is simply connected but not contractible. This is certainly plausible but we need a little more than the fundamental group to prove it.

Bibliographic notes. Most of the material for this section is taken from the treatment of the fundamental group in the text by Massey [1]. This group has been introduced by Poincaré in 1895 [2]. In a sequence of papers, Poincaré introduced seminal ideas which later led to the development of algebraic topology as a field in mathematics. In one of the papers, Poincaré ventured the now famous conjecture that every compact, simply connected 3-manifold without boundary is homeomorphic to the 3-sphere, which has only recently been settled in the affirmative by Perelman.

[1] W. S. Massey. *Algebraic Topology: an Introduction.* Springer-Verlag, New York, 1967.

[2] H. Poincaré. Analysis situs. *J. Ecole Polytechn.* **1** (1895), 1–121.

[3] H. Poincaré. Cinquième complément a l'analysis situs. *Rend. Circ. Mat. Palermo* **18** (1904), 45–110.