III.1 Simplicial Complexes

There are many ways to represent a topological space, one being a collection of simplices that are glued to each other in a structured manner. Such a collection can easily grow large but all its elements are simple. This is not so convenient for hand-calculations but close to ideal for computer implementations. In this book, we use simplicial complexes as the primary representation of topology.

Simplices. Let $u_0, u_1, \ldots, u_k$ be points in $\mathbb{R}^d$. A point $x = \sum_{i=0}^{k} \lambda_i u_i$ is an *affine combination* of the u_i if the λ_i sum to 1. The *affine hull* is the set of affine combinations. It is a *k-plane* if the $k+1$ points are *affinely independent* by which we mean that any two affine combinations, $x = \sum \lambda_i u_i$ and $y = \sum \mu_i u_i$, are the same iff $\lambda_i = \mu_i$ for all i. The $k+1$ points are affinely independent iff the k vectors $u_i - u_0$, for $1 \leq i \leq k$, are linearly independent. In $\mathbb{R}^d$ we can have at most d linearly independent vectors and therefore at most $d+1$ affinely independent points.

An affine combination $x = \sum \lambda_i u_i$ is a *convex combination* if all λ_i are non-negative. The *convex hull* is the set of convex combinations. A *k-simplex* is the convex hull of $k+1$ affinely independent points, $\sigma = \operatorname{conv}\{u_0, u_1, \ldots, u_k\}$. We sometimes say the u_i *span* σ. Its *dimension* is $\dim \sigma = k$. We use special names of the first few dimensions, *vertex* for 0-simplex, *edge* for 1-simplex, *triangle* for 2-simplex, and *tetrahedron* for 3-simplex; see Figure III.1. Any subset of

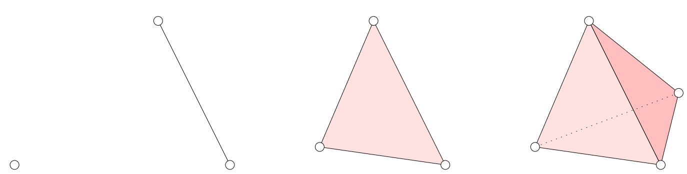

Figure III.1: From left to right: a vertex, an edge, a triangle, and a tetrahedron.

affinely independent points is again affinely independent and therefore also defines a simplex. A *face* of σ is the convex hull of a non-empty subset of the u_i and it is *proper* if the subset is not the entire set. We sometimes write $\tau \leq \sigma$ if τ is a face and $\tau < \sigma$ if it is a proper face of σ. Since a set of size $k+1$ has 2^{k+1} subsets, including the empty set, σ has $2^{k+1} - 1$ faces, all of which are proper except for σ itself. The *boundary* of σ, denoted as $\operatorname{bd} \sigma$, is the union of all proper faces, and the *interior* is everything else, $\operatorname{int} \sigma = \sigma - \operatorname{bd} \sigma$. A point $x \in \sigma$ belongs to $\operatorname{int} \sigma$ iff all its coefficients λ_i are positive. It follows that every point $x \in \sigma$ belongs to the interior of exactly one face, namely the one spanned by the points u_i that correspond to positive coefficients λ_i.

Simplicial complexes. We are interested in sets of simplices that are closed under taking faces and that have no improper intersections.

DEFINITION. A *simplicial complex* is a finite collection of simplices K such that $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$, and $\sigma, \sigma_0 \in K$ implies $\sigma \cap \sigma_0$ is either empty or a face of both.

The *dimension* of K is the maximum dimension of any of its simplices. The *underlying space*, denotes as $|K|$, is the union of its simplices together with the topology inherited from $\mathbb{R}^d$. A *polyhedron* is the underlying space of a simplicial complex. A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex K together with a homeomorphism between $\mathbb{X}$ and $|K|$. The topological space is *triangulable* if it has a triangulation. A *subcomplex* of K is a simplicial complex $L \subseteq K$. It is *full* if it contains all simplices in K spanned by vertices in L. A particular subcomplex is the j*-skeleton* consisting of all simplices of dimension j or less, $K^{(j)} = \{\sigma \in K \mid \dim \sigma \leq j\}$. The 0-skeleton is also referred to as the *vertex set*, $\operatorname{Vert} K = K^{(0)}$. Skeleta are generally not full. A subset of a simplicial complex useful in talking about local neighborhoods is the *star* of a simplex τ consisting of all simplices that have τ as a face, $\operatorname{St} \tau = \{\sigma \in K \mid \tau \leq \sigma\}$. Generally, the star is not closed under taking faces. We can make it into a complex by adding all missing faces. The result is the *closed star*, $\overline{\operatorname{St}}\, \tau$, which is the smallest subcomplex that contains the star. The *link* consists of all simplices in the closed star that are disjoint from τ, $\operatorname{Lk} \tau = \{\upsilon \in \overline{\operatorname{St}}\, \tau \mid \upsilon \cap \tau = \emptyset\}$. It τ is a vertex then the link is just the difference between the closed star and the star. More generally, it is the closed star minus the stars of all faces of τ. For example if K triangulates a 2-manifold without boundary then the link of an edge is a pair of points, a 0-sphere, and the link of a vertex is a cycle of edges and vertices, a 1-sphere.

Abstract simplicial complex. It is often easier to construct a complex abstractly and to worry abut how to put it into Euclidean space later.

DEFINITION. An *abstract simplicial complex* is a finite collection of sets A such that $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$.

The sets in A are its *simplices*. The *dimension* of a simplex is $\dim \alpha = \operatorname{card} \alpha - 1$ and the dimension of the complex is the maximum dimension of any of its simplices. A *face* of α is a non-empty subset $\beta \subseteq \alpha$, which is *proper* if $\beta \neq \alpha$. The *vertex set* is the union of all simplices, $\operatorname{Vert} A = \bigcup A = \bigcup_{\alpha \in A} \alpha$. A *subcomplex* is an abstract simplicial complex $B \subseteq A$. Two abstract simplicial complexes are *isomorphic* if there is a bijection $b : \operatorname{Vert} A \to \operatorname{Vert} B$ such that $\alpha \in A$ iff $b(\alpha) \in B$. The largest abstract simplicial complex with a vertex set of size n has cardinality $2^n - 1$. Given a (geometric) simplicial complex K, we can construct an abstract simplicial complex A by throwing away all simplices and retaining only their sets of vertices. We call A a *vertex scheme* of K. Symmetrically, we call K a *geometric realization* of A but also of every abstract simplicial complex isomorphic to A. Constructing geometric realizations is surprisingly easy if the dimension of the ambient space is sufficiently high.

Geometric Realization Theorem. An abstract simplicial complex of dimension d has a geometric realization in $\mathbb{R}^{2d+1}$.

Proof. Let $f : \operatorname{Vert} A \to \mathbb{R}^{2d+1}$ be an injection whose image is a set of points in general position. Specifically, any $2d + 2$ or fewer of the points are affinely independent. Let α and α_0 be simplices in A with $k = \dim \alpha$ and $k_0 = \dim \alpha_0$. The union of the two has size $\operatorname{card}(\alpha \cup \alpha_0) = \operatorname{card} \alpha + \operatorname{card} \alpha_0 - \operatorname{card}(\alpha \cap \alpha_0) \leq k + k_0 + 2 \leq 2d + 2$. The points in $\alpha \cup \alpha_0$ are therefore affinely independent, which implies that every convex combination x of points in $\alpha \cup \alpha_0$ is unique. Hence x belongs to $\sigma = \operatorname{conv} f(\alpha)$ as well as to $\sigma_0 = \operatorname{conv} f(\alpha_0)$ iff x is a convex combination of $\alpha \cap \alpha_0$. This implies that the intersection of σ and σ_0 is either empty of the simplex $\operatorname{conv} f(\alpha \cap \alpha_0)$, as required. ◻

Simplicial maps. Let K be a simplicial complex with vertices $u_0, u_1, \ldots, u_n$. Every point $x \in |K|$ belongs to the interior of exactly one simplex in K. Letting $\sigma = \operatorname{conv}\{u_0, u_1, \ldots, u_k\}$ be this simplex, we have $x = \sum_{i=0}^{k} \lambda_i u_i$ with $\sum_{i=0}^{k} \lambda_i = 1$ and $\lambda_i > 0$ for all i. Setting $b_i(x) = \lambda_i$ for $0 \leq i \leq k$ and $b_i(x) = 0$ for $k+1 \leq i \leq n$ we have $x = \sum_{i=0}^{n} b_i(x) u_i$ and we call the $b_i(x)$ the *barycentric coordinates* of x in K. We use barycentric coordinates to construct continuous maps.

Definition. A *vertex map* is a function $\varphi : \operatorname{Vert} K \to \operatorname{Vert} L$ with the property that the vertices of every simplex in K map to vertices of a simplex in L. Then φ can be extended to a continuous map $f : |K| \to |L|$ defined by

$$f(x) = \sum_{i=0}^{n} b_i(x) \varphi(u_i),$$

the *simplicial map* induced by φ.

There is an alternative way to think of this construction. Fix a vertex u_j and consider the map $b_j : |K| \to \mathbb{R}$ which maps each point x to its j-th barycentric coordinate. The graph of this map has the shape of a hat, increasing from zero on and outside the link to one at u_j. The map b_j is continuous and is sometimes referred to as a basis function. The simplicial map is thus the weighted sum of the $n+1$ basis functions. To emphasize that the simplicial map is linear on every simplex we usually drop the underlying space from the notation and write $f : K \to L$. As an example we consider the simplicial map $f : [0,1]^2 \to \mathbb{T}^2$ illustrated in Figure III.2. Given the vertex map, the simplicial

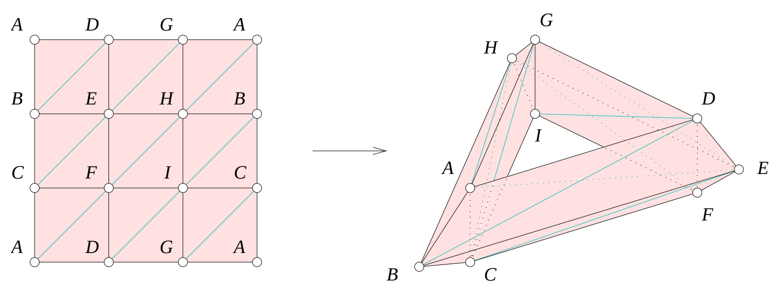

Figure III.2: A vertex map and its induced simplicial map from the square to the torus.

map is unique and glues the simplices of the triangulation of the square to obtain a triangulation of the torus. If the vertex map $\varphi : \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ is bijective and $\varphi^{-1} : \mathrm{Vert}\, L \to \mathrm{Vert}\, K$ is also a vertex map then the induced simplicial map f is a homeomorphism. In this case we call f a *simplicial homeomorphism* or an *isomorphism* between K and L.

Subdivisions. A simplicial complex L is a *subdivision* of another simplicial complex K if $|L| = |K|$ and every simplex in L is contained in a simplex in K. There are many ways to construct subdivisions. A particular one is the *barycentric subdivision*, $L = \mathrm{Sd}K$, illustrated in Figure III.3. A crucial concept in its construction is the *barycenter* of a simplex which is the average of its vertices. We proceed by induction over the dimension. To get started, the barycentric subdivision of the 0-skeleton is the same, $\mathrm{Sd}K^{(0)} = K^{(0)}$. Assuming we have the barycentric subdivision of $K^{(j-1)}$, we construct $\mathrm{Sd}K^{(j)}$ by adding the barycenter of every j-simplex as a new vertex and connecting it to the simplices that subdivide the boundary of the j-simplex.

The *diameter* of a set in Euclidean space is the supremum over the distances between its points. Since the simplices of K are point sets in Euclidean space

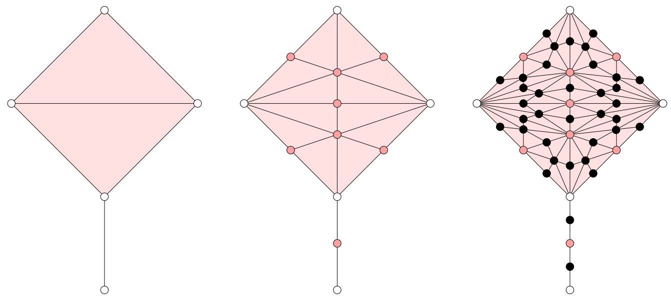

Figure III.3: Left: a simplicial complex consisting of two triangles, six edges, and five vertices. Middle and right: its first two barycentric subdivisions.

their diameters are well defined. The *mesh* of K is the maximum diameter of any simplex or, equivalently, the length of its longest edge.

MESH LEMMA. Letting δ be the mesh of the d-dimensional simplicial complex K, the mesh of $\mathrm{Sd}K$ is at most $\frac{d}{d+1}\delta$.

PROOF. Let τ and υ be complementary faces of a simplex $\sigma \in K$, that is, $\tau \cap \upsilon = \emptyset$ and $\dim \tau = \dim \upsilon = \dim \sigma - 1$. The line segment connecting the barycenters of τ and υ has length at most δ, and it splits into two edges in $\mathrm{Sd}K$, in proportions $1 + \dim \upsilon$ to $1 + \dim \tau$. The fraction of length is therefore between $\frac{1}{k+1}$ and $\frac{k}{k+1}$, where $k = \dim \sigma$. Both edges have therefore length at most $\frac{d}{d+1}$ times δ. ⊒

By the Mesh Lemma we can make the diameters of the simplices as small as we like by iterating the subdivision operation. For $n \geq 1$, the *n-th barycentric subdivision* of K is $\mathrm{Sd}^n K = \mathrm{Sd}(\mathrm{Sd}^{n-1} K)$. As n goes to infinity the mesh of $\mathrm{Sd}^n K$ goes to zero.

Simplicial approximations. It is sometimes convenient to think of a vertex star as an open set of points. Formally, we define $N(u) = \bigcup_{\sigma \in \mathrm{St}\, u} \mathrm{int}\, \sigma$. Let K and L be simplicial complexes. A continuous map $g : |K| \to |L|$ satisfies the *star condition* if the image of every vertex star in K is contained in a vertex star in L, that is, for each vertex $u \in K$ there is a vertex $v \in L$ such that $g(N(u)) \subseteq N(v)$. Let $\varphi : \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ map u to the vertex $\varphi(u) = v$ that exists by the star condition. To understand this new function, we take a point x in the interior of a simplex σ in K. Its image, $g(x)$, lies in the interior of a unique simplex τ in L. By definition of star condition, each vertex u of σ maps to a vertex $\varphi(u)$ of τ. Hence φ is a vertex map and induces a simplicial map $f : K \to L$. This map satisfies the condition of an *simplicial approximation* of g, namely $g(N(u)) \subseteq N(f(u))$ for each vertex u of K. We illustrated the definitions in Figure III.4. The image we have in mind is that g and f are not

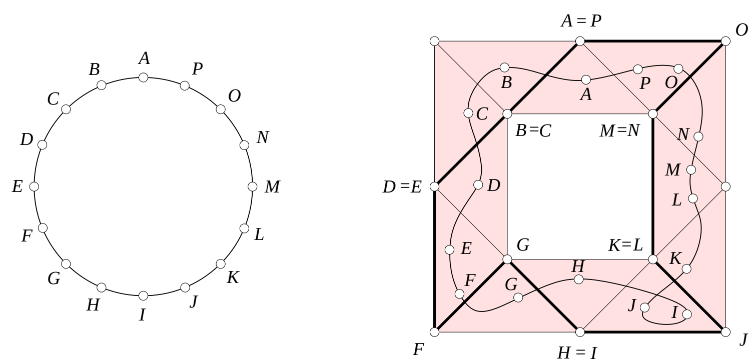

Figure III.4: The circle on the left is mapped into the closed annulus by a continuous map and a simplicial approximation of that map. Corresponding vertices are labeled by the same letter.

too different. In particular, $g(x)$ and $f(x)$ belong to a common simplex in L for every $x \in |K|$. Given a continuous map $g : |K| \to |L|$, it is plausible that we can subdivide K sufficiently finely so that a simplicial approximation exists. To be sure we prove this fact.

Simplicial Approximation Theorem. If $g : |K| \to |L|$ is continuous then there is a sufficiently large integer n such that g has a simplicial approximation $f : \mathrm{Sd}^n K \to L$.

Proof. Cover $|K|$ with open sets of the form $g^{-1}(N(v))$, $v \in \mathrm{Vert}\, L$. Since $|K|$ is compact there is a positive real number λ such that any set of diameter less than λ is contained in one of the sets in the open cover. Choose n such that each simplex in $\mathrm{Sd}^n K$ has diameter less than half of λ. Then each star in K has diameter less than λ implying it lies in one of the sets $g^{-1}(N(v))$. Hence g satisfies the star condition implying the existence of a simplicial approximation. ⊔⊓

Bibliographic notes. The terminology we use for abstract and geometric simplicial complexes follows the one in Munkres [3]. We have seen that $2d + 1$

dimensions suffice for the geometric realization of any d-dimensional abstract simplicial complex. Complexes that require that many dimensions have been described by Flores [1] and van Kampen [5]. An example of such a complex is the d-skeleton of the $(2d+2)$-simplex, which does not embed in $\mathbb{R}^{2d}$. For $d = 1$ this is the complete graph of five vertices, which does not embed in the plane as discussed in Chapter I.

A stronger version of the Simplicial Approximation Theorem played an important role in the development of combinatorial topology during the first half of the twentieth century. Known as the Hauptvermutung (German for "main conjecture"), it claimed that any two simplicial complexes that triangulate the same topological space have isomorphic subdivisions. This turned out to be correct for simplicial complexes of dimension 2 and 3 but not higher. The first counterexample found by Milnor was a simplicial complex of dimension 7 [2]. We refer to the book edited by Ranicki [4] for further information on the topic.

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