III.3 Delaunay Complexes

In this section, we introduce more elaborate geometric constructions to limit the dimension of the simplices we get from nerves to the dimension of the ambient space, at least for generic input.

Inversion. Recall that $\mathbb{S}^d$ is the d-dimensional sphere with center at the origin and unit radius in $\mathbb{R}^{d+1}$. To invert $\mathbb{R}^{d+1}$, we map each point $x \neq 0$ to the point on the same half-line whose distance from the origin is one over the distance of x from 0. More formally, the *inversion* is the map $\iota = \iota_{0,1} : \mathbb{R}^{d+1} - \{0\} \to \mathbb{R}^{d+1} - \{0\}$ defined by $\iota(x) = x/\|x\|^2$. Clearly, $\iota(\iota(x)) = x$. The inversion maps points inside $\mathbb{S}^d$ to points outside $\mathbb{S}^d$ and vice versa. Points on $\mathbb{S}^d$ remain fixed. For a point inside $\mathbb{S}^d$ we can construct its image, $x' = \iota(x)$, by drawing right-angled triangles. First we get $0xp$ with $p \in \mathbb{S}^d$ and the right angle at x. Second we draw $0px'$ with the right angle at p. The angle at 0 is the same in both so the two triangles are similar. Hence, $\|x\| : \|p\| = \|p\| : \|x'\|$ which implies $\|x\|\|x'\| = \|p\|^2 = 1$, as required.

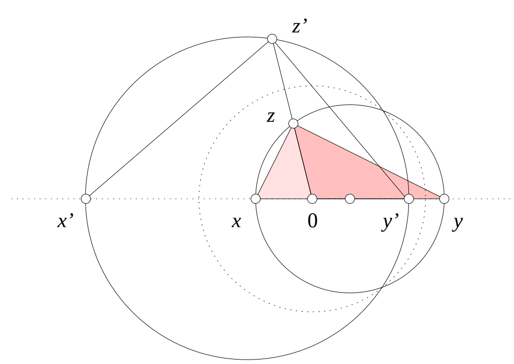

Figure III.9: As z sweeps out the circle passing through x and y its inverse image, z' sweeps out the circle passing through x' and y'.

A useful property is that inversion preserves spheres. In other words, the set of images of points on a sphere form another sphere. However, the image of the center of the first sphere is generally not the center of the second sphere. To prove this result consider a sphere that does not pass through 0, as in Figure III.9. Draw the line passing through 0 and the center; it intersects the sphere in points x and y, which we invert to get points $x' = \iota(x)$ and $y' = \iota(y)$. Let z be another point on the sphere and $z' = \iota(z)$ its inverse. Then $\|x\|\|x'\| = \|z\|\|z'\|$

and hence the triangles $0xz$ and $0z'x'$ are similar. By the same token, $0yz$ and $0z'y'$ are similar. But xyz has a right angle at z implying the angles at x' and y' inside $x'y'z'$ add up to a right angle. It follows that $x'y'z'$ has a right angle at z'. As z travels on the sphere with diameter xy the image z' travels on the sphere with diameter $x'y'$. What happens when the sphere passes through the origin, say $0 = x$? Then the triangle $0y'z'$ has a right angle at y'. Equivalently, the image of the sphere is the plane normal to the vector y and passing through the point y'. It thus makes sense to think of a plane as a sphere with the distinction that it passes through the point at infinity.

Stereographic projection. The inversion of $(d+1)$-dimensional space can be defined relative to any center $z \in \mathbb{R}^{d+1}$ and any radius $r > 0$. We consider the special case in which the center is the point $N = (0, \ldots, 0, 1)$, the north-pole of $\mathbb{S}^d$, and the radius is $r = \sqrt{2}$, the Euclidean distance between the north-pole and the equator. The image of $\mathbb{S}^d$ is the d-plane of points with vanishing $(d+1)$-st coordinates, which we denote as $\mathbb{R}^d$. The *stereographic projection* is the restriction of this particular inversion to the unit sphere, that is, $\varsigma : \mathbb{S}^d - \{N\} \to \mathbb{R}^d$ defined by $\varsigma(x) = \iota_{N,\sqrt{2}}(x)$; see Figure III.10. A useful

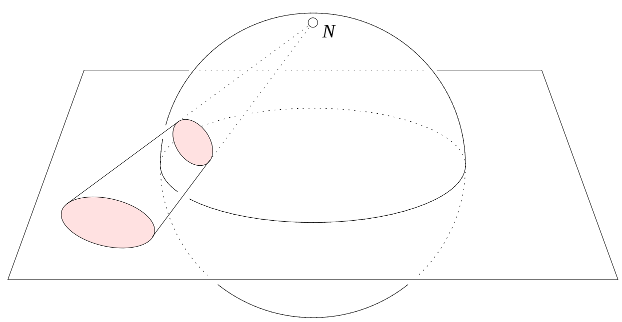

Figure III.10: The stereographic projection maps a circle on the unit sphere to a circle in the plane. If the circle on the sphere passes through the north-pole then its image is a line, a circle that passes through the point at infinity.

fact is that ς maps every $(d-1)$-sphere in $\mathbb{S}^d$ to a $(d-1)$-sphere in $\mathbb{R}^d$, and vice versa. Indeed, every $(d-1)$-sphere is the intersection of $\mathbb{S}^d$ with another d-sphere. Its image is therefore the intersection of $\mathbb{R}^d$ with the image of the d-sphere, which is another d-sphere. As before, we consider a plane as a special sphere that passes through the point at infinity. The preimage of that point is the north-pole, so the preimage of a plane in $\mathbb{R}^d$ is a sphere that passes through the north-pole.

Voronoi diagram. We will use the stereographic projection and the more general inversion to elucidate the construction of a particular simplicial complex from a finite set $S \subseteq \mathbb{R}^d$. The *Voronoi cell* of a point p in S is the set of points for which p is the closest,

$$V_p = \{x \in \mathbb{R}^d \mid \|x - p\| \leq \|x - q\|, \; q \in S\}.$$

It is the intersection of half-spaces of points at least as close to p as to q, over all points q in S. In other words, V_p is a convex polyhedron in $\mathbb{R}^d$. Any two Voronoi cells meet at most in a common piece of their boundary, and together the Voronoi cells cover the entire space, as illustrated in Figure III.11.

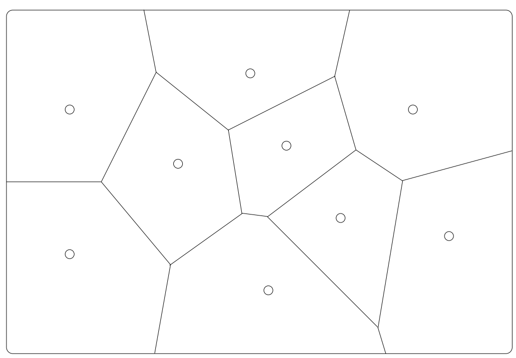

Figure III.11: The Voronoi diagram of nine points in the plane. By definition, each vertex of the diagram is equally far from the points that generate the incident Voronoi cells and further from all other points in S.

Power diagrams. In some circumstances, it is convenient to generalize the concept and consider points with real weights. Writing w_p for the weight of the point $p \in S$, the *weighted square distance* or *power* of a point $x \in \mathbb{R}^d$ from p is $\pi_p(x) = \|x - p\|^2 - w_p$. For positive weight we can interpret the weighted point as the sphere with center p and square radius w_p. For a point x outside that sphere the power is positive and equal to the square length of a tangent line segment from x to the sphere. For x on the sphere the power is zero, and for x inside the power is negative. The *bisector* of two weighted points is the set of points with equal power from both. Just like in the unweighted case, the bisector is a plane normal to the line connecting the two points, except that it is not necessarily halfway between them; see Figure III.12. In complete analogy to the unweighted case, we define the *weighted Voronoi* or *power cell*

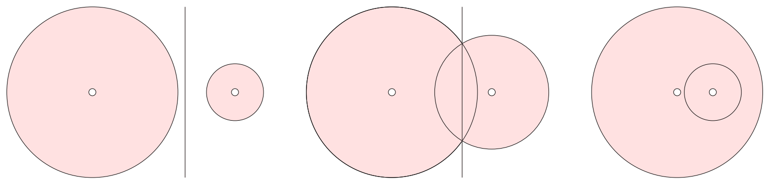

Figure III.12: The bisector of two weighted points. From left to right: two circles side by side, two intersecting circles, and two nested circles.

as the set of points for which p minimizes the power,

$$V_p = \{x \in \mathbb{R}^d \mid \pi_p(x) \leq \pi_q(x),\ q \in S\}.$$

Diagrams consisting of weighted or unweighted Voronoi cells look rather similar and are difficult to distinguish by eye, unless the locations of the generating points are marked. In the unweighted case each point has a non-empty Voronoi cell while in the weighted case a cell can be empty.

Lifting. We get a different and perhaps more illuminating view of Voronoi cells by lifting them to one higher dimension. Let S be a finite set of points in $\mathbb{R}^d$, as before, but draw them in $\mathbb{R}^{d+1}$, adding zeros as $(d+1)$-st coordinates. Map each point p in S to $\mathbb{S}^d$ using inverse stereographic projection, and let h_p be the d-plane tangent to $\mathbb{S}^d$ touching the sphere in the point $\varsigma^{-1}(p)$, as illustrated in Figure III.13. Using inversion, we now map each d-plane h_p to the d-sphere $s_p = \iota(h_p)$. It passes through the north-pole and is tangent to $\mathbb{R}^d$, the preimage of $\mathbb{S}^d$. The arrangements of planes and of spheres are closely related to the diagram of Voronoi cells. We focus on the spheres first.

Claim A. A point $x \in \mathbb{R}^d$ belongs to the Voronoi cell of $p \in S$ iff the first intersection of the directed line segment from x to N is with the d-sphere s_p.

Proof. Interpret the sphere s_p as a weighted point, namely the center with weight equal to the square radius. The power of a point x is the square length of a tangent line segment, which is equal to $\|x-p\|^2$ if $x \in \mathbb{R}^d$. It follows that the weighted Voronoi cell of the weighted center intersect $\mathbb{R}^d$ in the Voronoi cell of p. The claim follows because all bisectors of the weighted points pass through N. ◻

Switching from spheres to planes we get a similar characterization of the Voronoi diagram in terms of tangent planes.

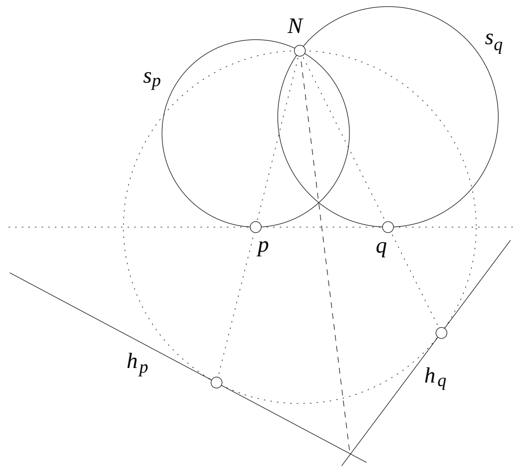

Figure III.13: We map p and q in $\mathbb{R}^1$ to lines h_p and h_q tangent to $\mathbb{S}^1$ and further to circles s_p and s_q passing through N and tangent to $\mathbb{R}^1$. The dashed line passes through the intersection of the two circles, the intersection of the two lines, as well as the midpoint between p and q.

Claim B. A point $x \in \mathbb{R}^d$ belongs to the Voronoi cell of $p \in S$ iff the first intersection of the directed line segment from N to x is with the d-plane h_p.

Delaunay triangulation. The *Delaunay complex* of a finite set $S \subseteq \mathbb{R}^d$ is isomorphic to the nerve of the collection of Voronoi cells,

$$\text{Delaunay} \quad = \quad \{\sigma \subseteq S \mid \bigcap_{p \in \sigma} V_p \neq \emptyset\}.$$

We say the set S is in general position if no $d+2$ of the points lie on a common $(d-1)$-sphere. The center of this sphere is in the common intersection of the Voronoi cells generated by the points on the sphere. The general position assumption thus implies that no $d+2$ Voronoi cells have a non-empty common intersection. Equivalently, the dimension of any simplex in the Delaunay complex is at most d. Assuming general position, we get a geometric realization by taking convex hulls of points in S. The result is often referred to as the *Delaunay triangulation* of S; see Figure III.14 for a two-dimensional example.

We conclude this section with a proof that taking convex hulls indeed gives a geometric realization of the Delaunay complex. Let $\iota(S)$ be the set of points

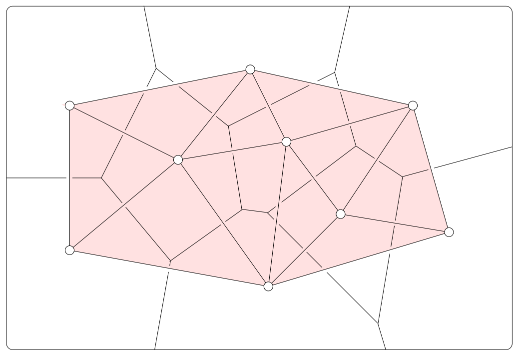

Figure III.14: The Delaunay triangulation superimposed on the Voronoi diagram. No four of the given points are cocircular implying that the Delaunay complex has simplices of dimension at most 2 and therefore a natural geometric realization in $\mathbb{R}^2$.

lifted to $\mathbb{S}^d$. Add N to this set and consider the convex hull of $\iota(S) \cup \{N\}$, which is a convex polytope P in $\mathbb{R}^{d+1}$.

Claim C. Let $S \subseteq \mathbb{R}^d$ be a finite set of points in general position. A point $x \in \mathbb{R}^d$ belongs to a d-simplex $\text{conv}\,\{p_0, p_1, \ldots, p_d\}$ in the Delaunay triangulation of S iff the open line segment from N to x intersects the boundary of P in the facet which is the convex hull of the points $\iota(p_0), \iota(p_1), \ldots, \iota(p_d)$.

Proof. The convex hull of $p_0, p_1, \ldots, p_d$ is a d-simplex in the Delaunay triangulation iff all other points of S lie outside the d-sphere that passes through the $d+1$ points. The preimage of that d-simplex under stereographic projection is a $(d-1)$-sphere in $\mathbb{S}^d$. Take the d-plane whose intersection with $\mathbb{S}^d$ is this $(d-1)$-sphere. The preimages of p_0 to p_d lie in this d-plane and those of all other points in S lie on the same side as N. Hence, the $d+1$ preimages form a facet of the convex hull iff the $d+1$ points form a simplex in the Delaunay triangulation. By construction, the d-simplex is the central projection of the facet from the north-pole. ⊒

Bibliographic notes. Voronoi diagrams are named after Georgy Voronoi [3] and Delaunay triangulations after Boris Delaunay (also Delone) [2]. Both structures have been studied centuries earlier and the oldest record we still have are notes from René Descartes. They are also immensely popular in many areas of science; see for example the survey article by Aurenhammer [1].

[1] F. Aurenhammer. Voronoi diagrams — a study of a fundamental geometric data structure. *ACM Comput. Surveys* **23** (1991), 345–405.

[2] B. Delaunay. Sur la sphère vide. *Izv. Akad. Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk* **7** (1934), 793–800.

[3] G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J. Reine Angew. Math.* **133** (1907), 97–178, and **134** (1908), 198–287.