V.3 Piecewise Linear Functions

In practical situations we rarely (or perhaps never) have smooth functions. Instead, we use other functions to approximate smooth functions. In this section, we use insights gained into the smooth case as a guide in our attempt to understand the piecewise linear case.

Lower star filtration. Let K be a simplicial complex with real values specified at all vertices. Using linear extension over the simplices, we obtain a *piecewise linear* (*PL*) *function* $f : K \to \mathbb{R}$. It is defined by $f(x) = \sum_i b_i(x) f(u_i)$, where the u_i are the vertices of K and the $b_i(x)$ are the barycentric coordinates of x; see Section III.1. To emphasize that f is linear on every simplex, we prefer the above notation over the more accurate $f : |K| \to \mathbb{R}$. It is convenient to assume that f is *generic* by which we mean that the vertices have distinct function values. We can then order the vertices by increasing function value as $f(u_1) < f(u_2) < \ldots < f(u_n)$. For each $0 \leq i \leq n$, we let K_i be the full subcomplex defined by the first i vertices. In other words, a simplex $\sigma \in K$ belongs to K_i iff each vertex u_j of σ satisfies $j \leq i$. Recall that the star of a vertex u_i is the set of cofaces of u_i in K. The *lower star* is the subset of simplices for which u_i is the vertex with maximum function value,

$$\mathrm{St}_{-} u_i \quad = \quad \{\sigma \in \mathrm{St}\, u_i \mid x \in \sigma \Rightarrow f(x) \leq f(u_i)\}.$$

By assumption of genericity, each simplex has a unique maximum vertex and thus belongs to a unique lower star. It follows that the lower stars partition K. Furthermore, K_i is the union of the first i lower stars. This motivates us to call the nested sequence of complexes $\emptyset = K_0 \subset K_1 \subset \ldots \subset K_n = K$ the *lower star filtration* of f. It will be useful to notice that the K_i are representative of the continuous family of sublevel sets. Specifically, for $f(u_i) \leq t < f(u_{i+1})$

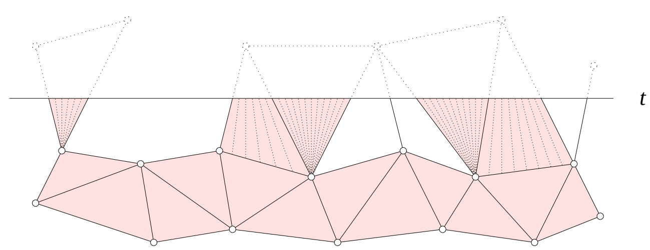

Figure V.8: We retract $|K|_t$ to K_i by shrinking the line segments decomposing the partial simplices from the top downward.

the sublevel set $|K|_t = f^{-1}(-\infty, t]$ has the same homotopy type as K_i. To prove this consider each simplex with at least one vertex in K_i and at least on in $K - K_i$. Write this simplex as a union of line segments connecting points on the maximal face in K_i with points on the maximal face in $K - K_i$, as shown in Figure V.8. The sublevel set $|K|_t$ contains only a fraction of each such line segment, namely the portion from the lower endpoint x in $|K_i|$ to the upper endpoint y with $f(y) = t$. To get a deformation retraction we let $y_\lambda = \lambda x + (1-\lambda)y$ be the upper endpoint at time λ. Going from time $\lambda = 0$ to $\lambda = 1$ proves that the sublevel set for t and K_i have the same homotopy type.

PL critical points. Let us study the change from one complex to the next in the lower star filtration in more detail. Recall that the link of a vertex is the set of simplices in the closed star that do not belong to the star. Similarly, the *lower link* is the collection of simplices in the lower star that do not belong to the lower star. Equivalently, it is the collection of simplices in the link whose vertices have smaller function value than u_i,

$$\mathrm{Lk}_- u_i \quad = \quad \{\sigma \in \mathrm{Lk}\, u_i \mid x \in \sigma \Rightarrow f(x) < f(u_i)\}.$$

When we go from K_{i-1} to K_i we attach the closed lower star of u_i, gluing it along the lower link to the complex K_{i-1}. Assume now that K triangulates a d-manifold. This restricts the possibilities dramatically since every vertex star is an open d-ball and every vertex link is a $(d-1)$-sphere. A few examples of lower stars and lower links in a 2-manifold are shown in Figure V.9. We classify

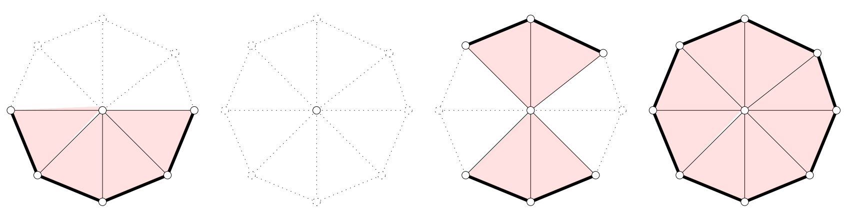

Figure V.9: From left to right: the lower star and lower link of a regular vertex, a minimum, a saddle, and a maximum.

the vertices using the reduced Betti numbers of their lower links. Recall that $\tilde\beta_0$ is one less than β_0, the number of components. The only exception to this rule is the empty lower link for which we have $\tilde\beta_0 = \beta_0 = 0$ and $\tilde\beta_{-1} = 1$. Table V.1 gives the reduced Betti numbers of the lower links in Figure V.9. We call u_i a *PL regular vertex* if its lower link is non-empty but homologically trivial and we call u_i a *simple PL critical vertex* of *index p* if its lower link has

	$\tilde{\beta}_{-1}$	$\tilde{\beta}_0$	$\tilde{\beta}_1$
regular	0	0	0
minimum	1	0	0
saddle	0	1	0
maximum	0	0	1

Table V.1: Classification of the vertices in a PL function on a 2-manifold.

the reduced homology of the $(p-1)$-sphere. In other words, the only non-zero reduced Betti number of a simple PL critical vertex of index p is $\tilde{\beta}_{p-1} = 1$.

Definition. A piecewise linear function $f : K \to \mathbb{R}$ on a manifold is a *PL Morse function* if (i) each vertex is either PL regular or simple PL critical and (ii) the function values of the vertices are distinct.

Unfolding. In contrast to the smooth case, PL Morse functions are not dense among the class of all PL functions. Equivalently, a PL function on a manifold may require a substantial perturbation before it becomes PL Morse. As an example consider the piecewise linear version of a monkey saddle displayed in Figure V.10. It is therefore not reasonable to assume a PL Morse function

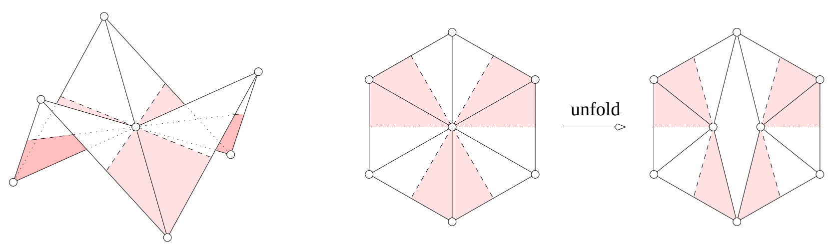

Figure V.10: Left: a PL monkey saddle of a height function. The areas of points lower than the center vertex are shaded. Right: the unfolding of the PL monkey saddle into two simple saddles.

as input, but we can sometimes alter the triangulation locally to make it into a PL Morse function. In the 2-manifold case, a k-fold saddle is defined by $\tilde{\beta}_0 = k$. We can split it into k simple saddles by introducing $k-1$ new vertices and assigning appropriate function values close to that of the original, k-fold saddle; see Figure V.10 for the case $k = 2$. It is less clear how to unfold possibly complicated PL critical points for higher-dimensional manifolds; see Section IX.8.

Alternating sum of indices. Let K be a triangulation of a d-manifold and $f : K \to \mathbb{R}$ a PL Morse function. It is not difficult to prove that the alternating sum of the simple PL critical points gives the Euler characteristic,

$$\chi(K) \quad = \quad \sum_u (-1)^{\mathrm{index}(u)}.$$

Since it is easy and instructive, we give an inductive proof of this equation. To go from K_{i-1} to K_i, we add the lower star of u_i. By the Euler-Poincaré Theorem, the Euler characteristic of the lower link $L = \mathrm{Lk}_- u_i$ is

$$\chi(L) \quad = \quad 1 + \sum_{p \geq 0} (-1)^{p-1} \tilde{\beta}_{p-1}(L),$$

which is 1 if u_i is PL regular and $1 + (-1)^{\mathrm{index}(u_i)-1}$ if u_i is PL critical. Each j-simplex in the lower star corresponds to a $(j-1)$-simplex in the lower link, except for the vertex u_i itself. Adding the lower star to the complex thus increases the Euler characteristic by $1 - \chi(L)$, which is zero for a PL regular point and $(-1)^{\mathrm{index}(u_i)}$ for a simple PL critical point. The claimed equation follows.

Mayer-Vietoris sequences. We prepare the proof of the complete set of Morse inequalities for PL Morse functions by introducing the Mayer-Vietoris sequence of a covering of a simplicial complex by two subcomplexes. Let $K = K' \cup S$ be the covering and note that the intersection of the two subcomplexes, $L = K' \cap S$, is also a subcomplex of K. The corresponding Mayer-Vietoris sequence is

$$\ldots \to \tilde{\mathsf{H}}_{p+1}(K) \xrightarrow{\varphi} \tilde{\mathsf{H}}_p(L) \xrightarrow{\psi} \tilde{\mathsf{H}}_p(K') \oplus \tilde{\mathsf{H}}_p(S) \to \tilde{\mathsf{H}}_p(K) \to \tilde{\mathsf{H}}_{p-1}(L) \to \ldots$$

It is *exact* which means that the image of every homomorphism is equal to the kernel of the next homomorphism in the sequence. This is not unlike the situation in a chain complex, except there the image can be smaller than the kernel and homology is a measure of that difference. We are interested in the reduced p-th homology group of L, letting φ and ψ be the maps that connect it to its predecessor and successor groups in the sequence. Let k_p be the rank of the kernel of ψ. Similarly, let k^p the rank of the cokernel of φ, that is, the rank of the quotient $\tilde{\mathsf{H}}_p(L)/\mathrm{im}\,\varphi$, which by exactness is $\tilde{\beta}_p(L) - k_p$. As illustrated in Figure V.11, exactness also implies that the rank of the image of ψ is k^p and the rank of $\tilde{\mathsf{H}}_{p+1}(K)/\ker\varphi$ is k_p.

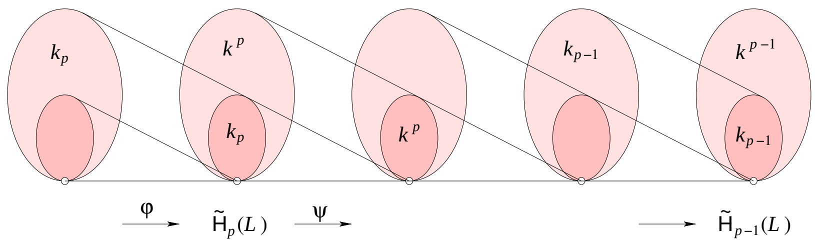

Figure V.11: A portion of the Mayer-Vietoris sequence. By exactness, the rank of the kernel of every map equals the rank of the cokernel of the preceding map.

PL Morse inequalities. We are now ready to state and prove the PL versions of the weak and strong Morse inequalities.

PL Morse Inequalities. Let K be a triangulation of a manifold of dimension d and $f : K \to \mathbb{R}$ a PL Morse function. Writing c_p for the number of index p PL critical points of f we have

(i) WEAK: $c_p \geq \beta_p(K)$ for all p;

(ii) STRONG: $\sum_{p=0}^{j}(-1)^{j-p}c_p \geq \sum_{p=0}^{j}(-1)^{j-p}\beta_p(K)$ for all j.

Proof. We prove the inequalities inductively, for each K_i. Note that K_i is the union of K_{i-1} and the closed lower star of u_i. We use the corresponding Mayer-Vietoris sequence, obtained by setting $K = K_i$, $K' = K_{i-1}$, $L = \mathrm{Lk}_- u_i$, and therefore $S = L \cup \mathrm{St}_- u_i$. Since S is the cone over a complex it is homologically trivial. Let φ and ψ be the maps as defined above, k_p the rank of the kernel of ψ, and k^p the rank of the cokernel of φ; refer to Figure V.11. Since S is trivial we have

$$\operatorname{rank} \tilde{\mathsf{H}}_p(K_i) \quad = \quad \operatorname{rank} \tilde{\mathsf{H}}_p(K_{i-1}) - k^p + k_{p-1}.$$

By definition of simple PL critical point we have $\tilde{\beta}_{p-1} = k_{p-1} + k^{p-1}$ for all p. If u_i is PL regular then $k_{p-1} = k^{p-1} = 0$ for all p and the ranks of the homology groups do not change. Similarly, none of the c_p changes so all Morse inequalities remain valid. If $\mathrm{index}(u_i) = p$ and $k_{p-1} = 1$ then both c_p and $\tilde{\beta}_p$ go up by one which maintains the validity of all Morse inequalities. On the other hand, if $\mathrm{index}(u_i) = p$ and $k^{p-1} = 1$ then c_p goes up and $\tilde{\beta}_{p-1}$ goes down. Since the two have opposite signs this maintains the validity of all Morse inequalities that contain both. The only strong Morse inequality that contains only one of the two terms is the one for $j = p - 1$. It contains the reduced Betti number with a plus sign so this inequality is also preserved. $\square$

We note that the strong Morse inequality for $j = d$ is actually an equality, namely the one we have proved above, before introducing the Mayer-Vietoris sequence. It contains both changing terms, in all cases, so there is never a chance that the two sides become different.

Bibliographic notes. Piecewise linear functions on polyhedral manifolds have already been studied by Banchoff [1]. He defines the index of a vertex as the Euler characteristic of its lower link. This is coarser than our definition but leads to similar results, in particular a short and elementary proof that the Euler characteristic is equal to the alternating sum of critical points. However, it does not lend itself to a natural generalization of the other Morse inequalities to non-Morse PL functions. Our classification of PL critical points in terms of reduced Betti numbers can be found in [3], where it is used to compute the PL analog of the Morse-Smale complex for 2-manifolds. There are industrial applications of these ideas to surface design and segmentation based on curvature approximating and other shape-sensitive functions in $\mathbb{R}^3$ [2].

[1] T. F. Banchoff. Critical points and curvature for embedded polyhedra. *J. Differential Geometry* **1** (1967), 245–256.

[2] H. Edelsbrunner. Surface tiling with differential topology (extended abstract of invited talk). *In* "Proc. 3rd Eurographics Sympos. Geom. Process., 2005", 9–11.

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