CPS 296.1
Auctions &
Combinatorial Auctions

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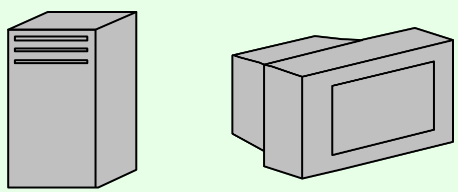
A few different 1-item auction mechanisms

- English auction:
  - Each bid must be higher than previous bid
  - Last bidder wins, pays last bid
- Japanese auction:
  - Price rises, bidders drop out when price is too high
  - Last bidder wins at price of last dropout
- Dutch auction:
  - Price drops until someone takes the item at that price
- Sealed-bid auctions (direct revelation mechanisms):
  - Each bidder submits a bid in an envelope
  - Auctioneer opens the envelopes, highest bid wins
    - First-price sealed-bid auction: winner pays own bid
    - Second-price sealed bid (or Vickrey) auction: winner pays second-highest bid

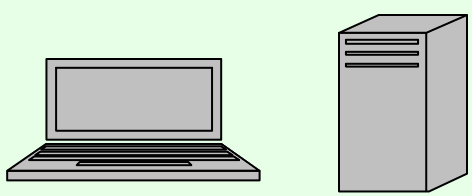
Complementarity and substitutability

- How valuable one item is to a bidder may depend on whether the bidder possesses another item

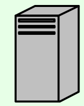
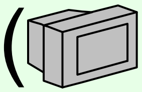
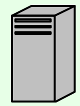
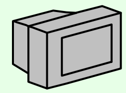
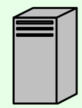
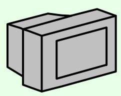
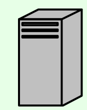
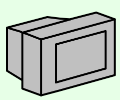
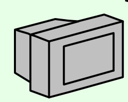
- Items a and b are complementary if $v(\{a, b\}) > v(\{a\}) + v(\{b\})$
- E.g.



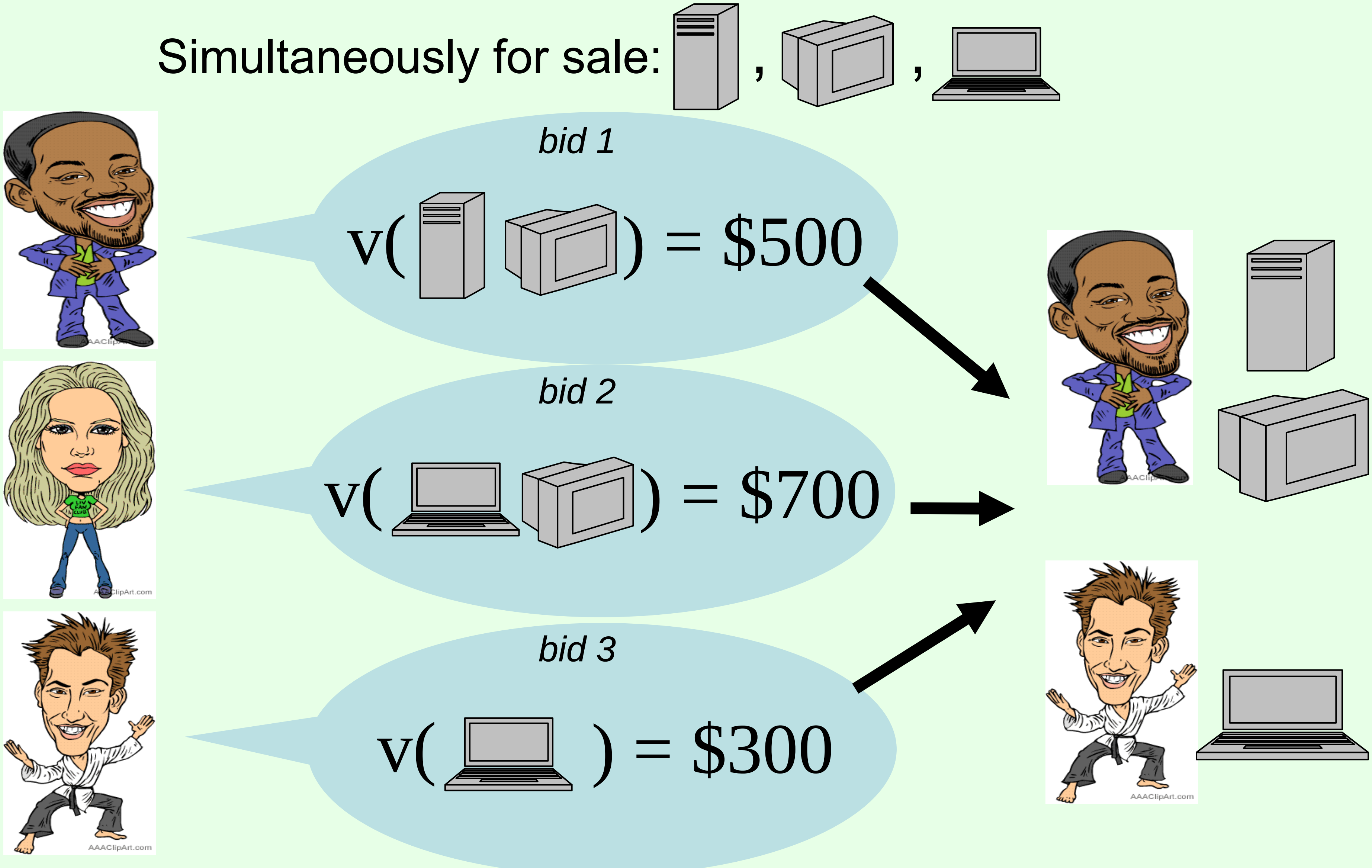
- Items a and b are substitutes if $v(\{a, b\}) < v(\{a\}) + v(\{b\})$
- E.g.



Inefficiency of sequential auctions

- Suppose your valuation function is v() = \$200, v() = \$100, v( ) = \$500
- Now suppose that there are two (say, Vickrey) auctions, the first one for  and the second one for 
- What should you bid in the first auction (for )?
- If you bid \$200, you may lose to a bidder who bids \$250, only to find out that you could have won  for \$200
- If you bid anything higher, you may pay more than \$200, only to find out that  sells for \$1000
- Sequential (and parallel) auctions are inefficient

Combinatorial auctions



used in truckload transportation, industrial procurement, radio spectrum allocation, …

The winner determination problem (WDP)

- Choose a subset A (the accepted bids) of the bids B,
- to maximize $\Sigma_{b \text{ in } A} v_b$,
- under the constraint that every item occurs at most once in A
  - This is assuming free disposal, i.e., not everything needs to be allocated

WDP example

- Items A, B, C, D, E
- Bids:
- ({A, C, D}, 7)
- ({B, E}, 7)
- ({C}, 3)
- ({A, B, C, E}, 9)
- ({D}, 4)
- ({A, B, C}, 5)
- ({B, D}, 5)

- *What's an optimal solution?*
- *How can we prove it is optimal?*

Price-based argument for optimality

- Items A, B, C, D, E
- Bids:
- ({A, C, D}, 7)
- ({B, E}, 7)
- ({C}, 3)
- ({A, B, C, E}, 9)
- ({D}, 4)
- ({A, B, C}, 5)
- ({B, D}, 5)

- Suppose we create the following “prices” for the items:
- p(A) = 0, p(B) = 7, p(C) = 3, p(D) = 4, p(E) = 0
- Every bid bids at most the sum of the prices of its items, so we can’t expect to get more than 14.

Price-based argument does not always give matching upper bound

- Items A, B, C
- Bids:
- ({A, B}, 2)
- ({B, C}, 2)
- ({A, C}, 2)

- Clearly can get at most 2
- If we want to set prices that sum to 2, there must exist two items whose prices sum to < 2
- But then there is a bid on those two items of value 2
  - (Can set prices that sum to 3, so that's an upper bound)

Should not be surprising, since it's an NP-hard problem and we don't expect short proofs for negative answers to NP-hard problems (we don't expect NP = coNP)

An integer program formulation

- x_b equals 1 if bid b is accepted, 0 if it is not
- maximize $\Sigma_b\, v_b x_b$
- subject to
  - for each item j, $\Sigma_{b:\ j \text{ in } b}\, x_b \leq 1$
- If each x_b can take any value in [0, 1], we say that bids can be partially accepted
- In this case, this is a linear program that can be solved in polynomial time
- This requires that
  - each item can be divided into fractions
  - if a bidder gets a fraction f of each of the items in his bundle, then this is worth the same fraction f of his value v_b for the bundle

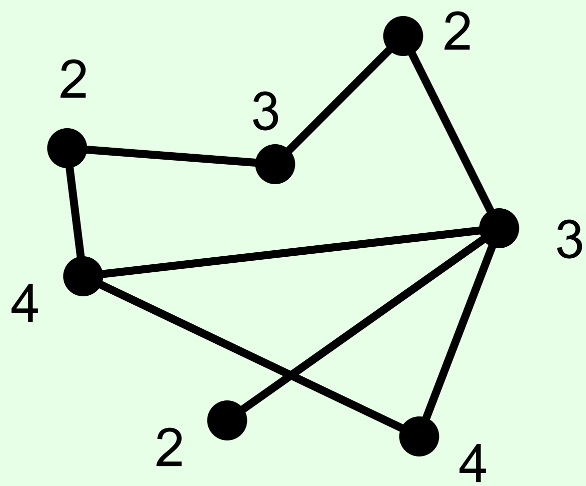
Price-based argument **does** always work for partially acceptable bids

- Items A, B, C
- Bids:
- ({A, B}, 2)
- ({B, C}, 2)
- ({A, C}, 2)

- Now can get 3, by accepting half of each bid
- Put a price of 1 on each item

General proof that with partially acceptable bids, prices always exist to give a matching upper bound is based on linear programming duality

Weighted independent set

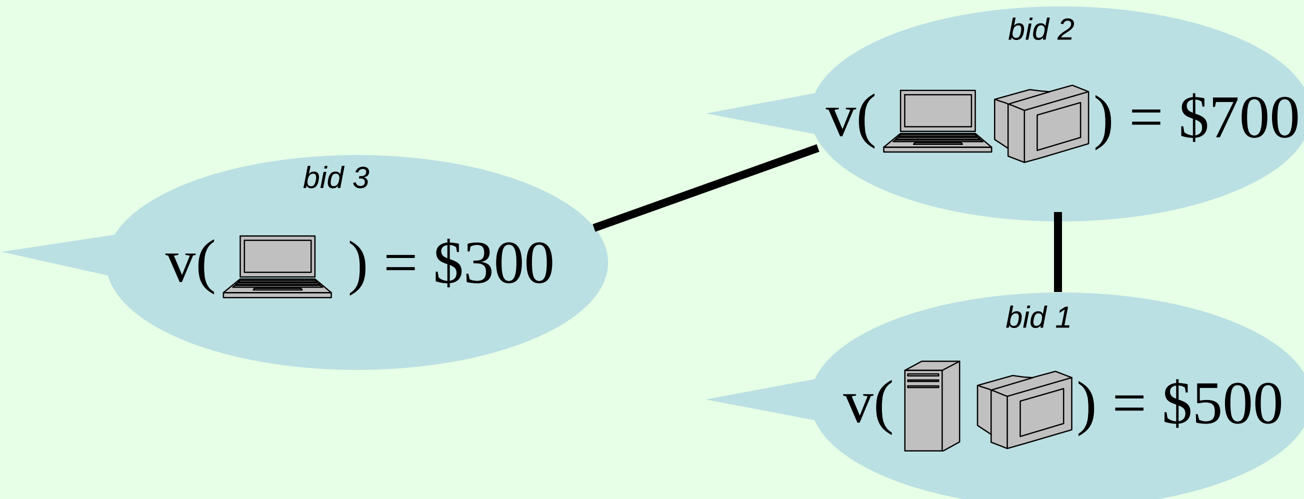


- Choose subset of the vertices with maximum total weight,
- Constraint: no two vertices can have an edge between them

- NP-hard (generalizes regular independent set)

The winner determination problem as a weighted independent set problem

- Each bid is a vertex
- Draw an edge between two vertices if they share an item



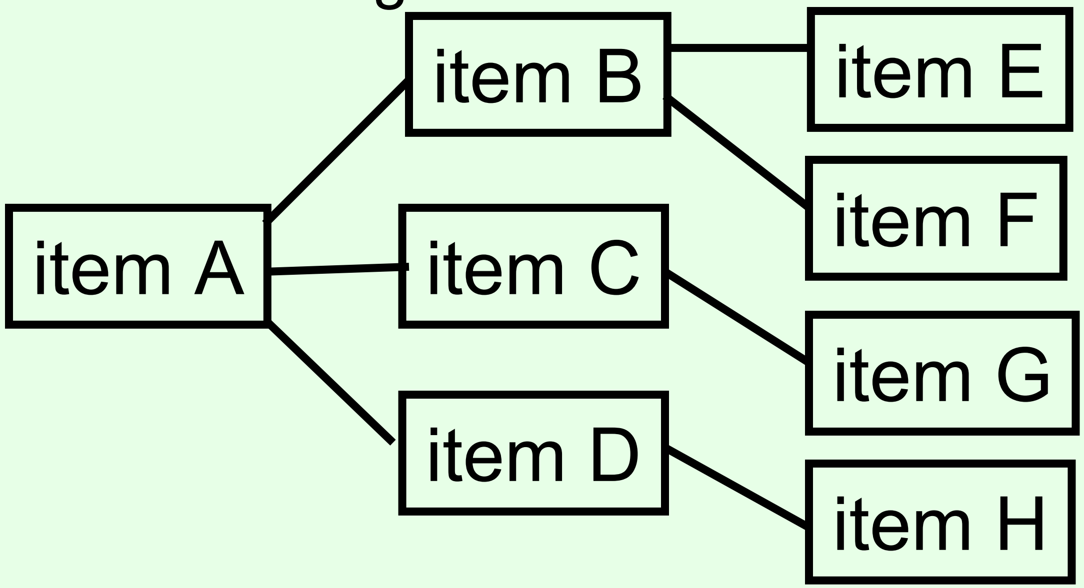
- Optimal allocation = maximum weight independent set
- Can model any weighted independent set instance as a CA winner determination problem (1 item per edge (or clique))
- Weighted independent set is NP-hard, even to solve approximately [Håstad 96] - hence, so is WDP
  - [Sandholm 02] noted that this inapproximability applies to the WDP

Dynamic programming approach to WDP [Rothkopf et al. 98]

- For every subset S of I, compute w(S) = the maximum total value that can be obtained when allocating only items in S
- Then, $w(S) = \max \{\max_i v_i(S), \max_{S'\text{: } S' \text{ is a subset of } S\text{, and there exists a bid on } S'} w(S') + w(S \setminus S')\}$
- Requires exponential time

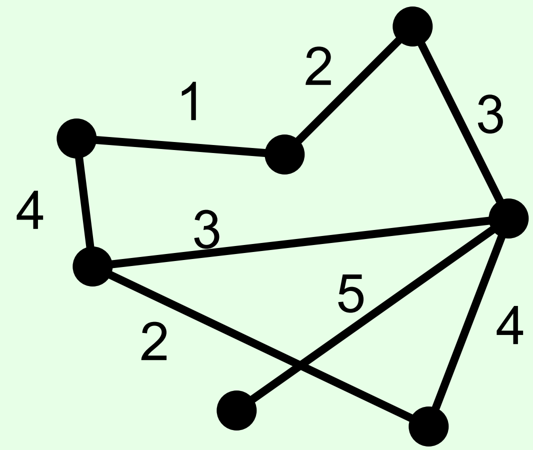
Bids on connected sets of items in a tree

- Suppose items are organized in a tree



- Suppose each bid is on a connected set of items
  - E.g. {A, B, C, G}, but not {A, B, G}
- Then the WDP can be solved in polynomial time (using dynamic programming) [Sandholm & Suri 03]
- Tree does not need to be given: can be constructed from the bids in polynomial time if it exists [Conitzer, Derryberry, Sandholm 04]
- More generally, WDP can also be solved in polynomial time for graphs of bounded treewidth [Conitzer, Derryberry, Sandholm 04]
  - Even further generalization given by [Gottlob, Greco 07]

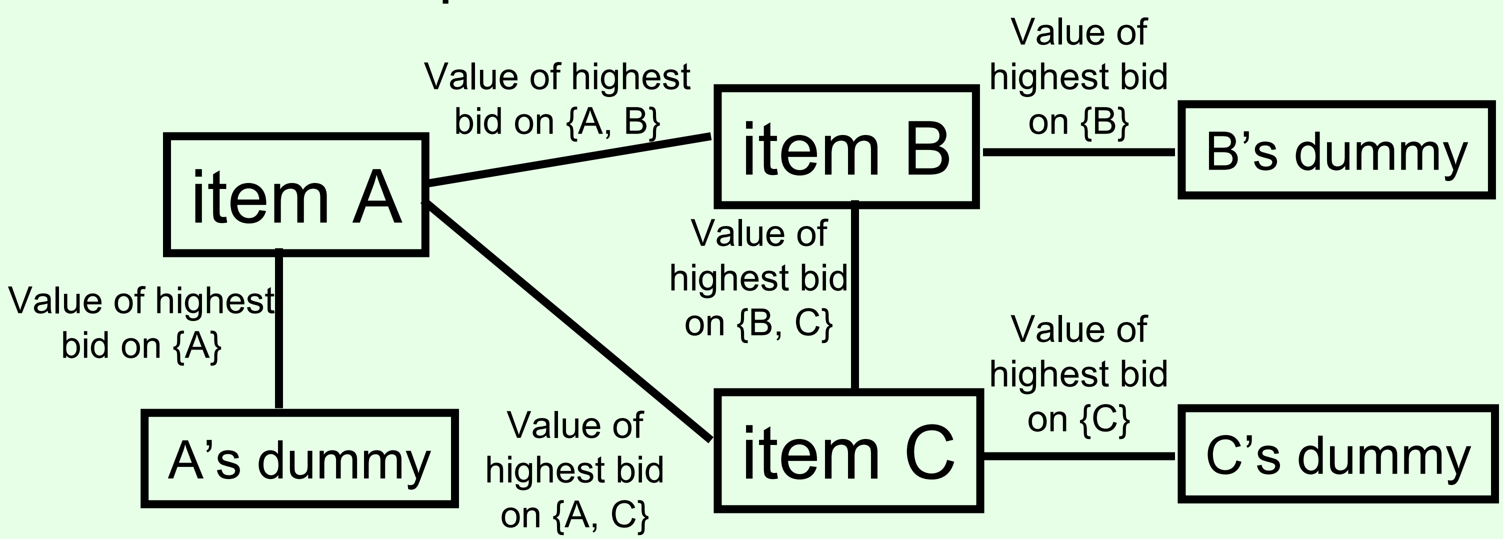
Maximum weighted matching
(not necessarily on bipartite graphs)



- Choose subset of the edges with maximum total weight,
- Constraint: no two edges can share a vertex
- Still solvable in polynomial time

Bids with few items [Rothkopf et al. 98]

- If each bid is on a bundle of at most two items,
- then the winner determination problem can be solved in polynomial time as a maximum weighted matching problem
  – 3-item example:



- If each bid is on a bundle of three items, then the winner determination problem is NP-hard again

Variants [Sandholm et al. 2002]: combinatorial reverse auction

- In a combinatorial reverse auction (CRA), the auctioneer seeks to buy a set of items, and bidders have values for the different bundles that they may sell the auctioneer
- minimize $\Sigma_b v_b x_b$
- subject to
  - for each item j, $\Sigma_{b:\, j \text{ in } b}\, x_b \geq 1$

WDP example (as CRA)

- Items A, B, C, D, E
- Bids:
- ({A, C, D}, 7)
- ({B, E}, 7)
- ({C}, 3)
- ({A, B, C, E}, 9)
- ({D}, 4)
- ({A, B, C}, 5)
- ({B, D}, 5)

Variants: multi-unit CAs/CRAs

- Multi-unit variants of CAs and CRAs: multiple units of the same item are for sale/to be bought, bidders can bid for multiple units
- Let q_{bj} be number of units of item j in bid b, q_j total number of units of j available/demanded

- maximize $\Sigma_b\, v_b x_b$
- subject to
  - for each item j, $\Sigma_b\, q_{bj} x_b \leq q_j$
- minimize $\Sigma_b\, v_b x_b$
- subject to
  - for each item j, $\Sigma_b\, q_{bj} x_b \geq q_j$

Multi-unit WDP example (as CA/CRA)

- Items: 3A, 2B, 4C, 1D, 3E
- Bids:
- ({1A, 1C, 1D}, 7)
- ({2B, 1E}, 7)
- ({2C}, 3)
- ({2A, 1B, 2C, 2E}, 9)
- ({2D}, 4)
- ({3A, 1B, 2C}, 5)
- ({2B, 2D}, 5)

Variants: (multi-unit) combinatorial exchanges

- Combinatorial exchange (CE): bidders can simultaneously be buyers and sellers
  - Example bid: "If I receive 3 units of A and -5 units of B (i.e., I have to give up 5 units of B), that is worth $100 to me."
- maximize $\Sigma_b v_b x_b$
- subject to
  - for each item j, $\Sigma_b q_{b,j} x_b \leq 0$

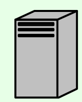
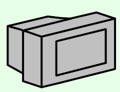
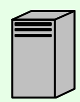
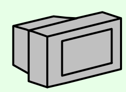
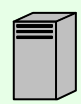
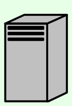
CE WDP example

- Bids:
- ({-1A, -1C, -1D}, -7)
- ({2B, 1E}, 7)
- ({2C}, 3)
- ({-2A, 1B, 2C, -2E}, 9)
- ({-2D}, -4)
- ({3A, -1B, -2C}, 5)
- ({-2B, 2D}, 0)

Variants: no free disposal

- Change all inequalities to equalities

(back to 1-unit CAs) Expressing valuation functions using bundle bids

- A bidder is single-minded if she only wants to win one particular bundle
  - Usually not the case
- But: one bidder may submit multiple bundle bids
- Consider again valuation function v() = \$200, v() = \$100, v( ) = \$500
- What bundle bids should one place?
- What about: v() = \$300, v() = \$200, v( ) = \$400?

Alternative approach: report entire valuation function

- I.e., every bidder i reports $v_i(S)$ for every subset S of I (the items)
- Winner determination problem:
- Allocate a subset S_i of I to each bidder i to maximize $\Sigma_i v_i(S_i)$ (under the constraint that for $i \neq j$, $S_i \cap S_j = \emptyset$)
  - This is assuming free disposal, i.e., not everything needs to be allocated

Exponentially many bundles

- In general, in a combinatorial auction with set of items I (|I| = m) for sale, a bidder could have a different valuation for every subset S of I
  - Implicit assumption: no externalities (bidder does not care what the other bidders win)
- Must a bidder communicate 2^m values?
  - Impractical
  - Also difficult for the bidder to evaluate every bundle
- Could require $v_i(\emptyset) = 0$
  - Does not help much
- Could require: if S is a superset of S', v(S) ≥ v(S') (free disposal)
  - Does not help in terms of number of values

Bidding languages

- Bidding language = a language for expressing valuation functions
- A good bidding language allows bidders to concisely express natural valuation functions
- Example: the OR bidding language [Rothkopf et al. 98, DeMartini et al. 99]
- Bundle-value pairs are ORed together, auctioneer may accept any number of these pairs (assuming no overlap in items)
- E.g. ({a}, 3) OR ({b, c}, 4) OR ({c, d}, 4) implies
  - A value of 3 for {a}
  - A value of 4 for {b, c, d}
  - A value of 7 for {a, b, c}
- Can we express the valuation function v({a, b}) = v({a}) = v({b}) = 1 using the OR bidding language?
- OR language is good for expressing complementarity, bad for expressing substitutability

XORs

- If we use XOR instead of OR, that means that only one of the bundle-value pairs can be accepted
- Can express any valuation function (simply XOR together all bundles)
- E.g. ({a}, 3) XOR ({b, c}, 4) XOR ({c, d}, 4) implies
  - A value of 3 for {a}
  - A value of 4 for {b, c, d}
  - A value of 4 for {a, b, c}
- Sometimes not very concise
- E.g. suppose that for any S, $v(S) = \Sigma_{s \text{ in } S} v(\{s\})$
  - How can this be expressed in the OR language?
  - What about the XOR language?
- Can also combine ORs and XORs to get benefits of both [Nisan 00, Sandholm 02]
- E.g. (({a}, 3) XOR ({b, c}, 4)) OR ({c, d}, 4) implies
  - A value of 4 for {a, b, c}
  - A value of 4 for {b, c, d}
  - A value of 7 for {a, c, d}

WDP and bidding languages

- Single-minded bidders bid on only one bundle
  - Valuation is v for any subset including that bundle, 0 otherwise
- If we can solve the WDP for single-minded bidders, we can also solve it for the OR language
  - Simply pretend that each bundle-value pair comes from a different bidder
- We can even use the same algorithm when XORs are added, using the following trick:
  - For bundle-value pairs that are XORed together, add a dummy item to them [Fujishima et al 99, Nisan 00]
  - E.g. ({a}, 3) XOR ({b, c}, 4) becomes ({a, $dummy_1$}, 3) OR ({b, c, $dummy_1$}, 4)
- So, we can focus on single-minded bids