CPS 296.1 Cooperative/coalitional game theory

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Cooperative/coalitional game theory

- There is a set of agents N
- Each subset (or coalition) S of agents can work together in various ways, leading to various utilities for the agents
- Cooperative/coalitional game theory studies which outcome will/should materialize
- Key criteria:
  - Stability: No coalition of agents should want to deviate from the solution and go their own way
  - Fairness: Agents should be rewarded for what they contribute to the group

- ("Cooperative game theory" is the standard name (distinguishing it from noncooperative game theory, which is what we have studied so far). However this is somewhat of a misnomer because agents still pursue their own interests. Hence some people prefer "coalitional game theory.")

Example

- Three agents {1, 2, 3} can go out for Indian, Chinese, or Japanese food
- $u_1(I) = u_2(C) = u_3(J) = 4$
- $u_1(C) = u_2(J) = u_3(I) = 2$
- $u_1(J) = u_2(I) = u_3(C) = 0$
- Each agent gets an additional unit of utility for each other agent that joins her
- Exception: going out alone always gives a total utility of 0
- If all agents go for Indian together, they get utilities (6, 2, 4)
- All going to Chinese gives (4, 6, 2), all going to Japanese gives (2, 4, 6)
- Hence, the utility possibility set for {1, 2, 3} is {(6, 2, 4), (4, 6, 2), (2, 4, 6)}
- For the coalition {1, 2}, the utility possibility set is {(5, 1), (3, 5), (1, 3)} (why?)

Stability & the core

- $u_1(I) = u_2(C) = u_3(J) = 4$
- $u_1(C) = u_2(J) = u_3(I) = 2$
- $u_1(J) = u_2(I) = u_3(C) = 0$
- $V(\{1, 2, 3\}) = \{(6, 2, 4), (4, 6, 2), (2, 4, 6)\}$
- $V(\{1, 2\}) = \{(5, 1), (3, 5), (1, 3)\}$
- Suppose the agents decide to all go for Japanese together, so they get (2, 4, 6)
- 1 and 2 would both prefer to break off and get Chinese together for (3, 5) – we say (2, 4, 6) is blocked by {1, 2}
  - Blocking only occurs if there is a way of breaking off that would make all members of the blocking coalition happier
- The core [Gillies 53] is the set of all outcomes (for the grand coalition N of all agents) that are blocked by no coalition
- In this example, the core is empty (why?)
- In a sense, there is no stable outcome

Transferable utility

- Now suppose that utility is transferable: you can give some of your utility to another agent in your coalition (e.g., by making a payment)
- Then, all that we need to specify is a value for each coalition, which is the maximum total utility for the coalition
  - Value function also known as characteristic function
- Any vector of utilities that sums to the value is possible
- Outcome is in the core if and only if: every coalition receives a total utility that is at least its value
  - For every coalition C, $v(C) \leq \Sigma_{i \text{ in } C} u(i)$
- In above example,
  - v({1, 2, 3}) = 12,
  - v({1, 2}) = v({1, 3}) = v({2, 3}) = 8,
  - v({1}) = v({2}) = v({3}) = 0
- Now the outcome (4, 4, 4) is possible; it is also in the core (why?) and in fact the unique outcome in the core (why?)

Emptiness & multiplicity

- Let us modify the above example so that agents receive no utility from being together (except being alone still gives 0)
  - v({1, 2, 3}) = 6,
  - v({1, 2}) = v({1, 3}) = v({2, 3}) = 6,
  - v({1}) = v({2}) = v({3}) = 0
- Now the core is empty!
- Conversely, suppose agents receive 2 units of utility for each other agent that joins
  - v({1, 2, 3}) = 18,
  - v({1, 2}) = v({1, 3}) = v({2, 3}) = 10,
  - v({1}) = v({2}) = v({3}) = 0
- Now lots of outcomes are in the core – (6, 6, 6), (5, 5, 8), …
- When is the core guaranteed to be nonempty?
- What about uniqueness?

Superadditivity

- v is superadditive if for all coalitions A, B with $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$
- Informally, the union of two coalitions can always act as if they were separate, so should be able to get at least what they would get if they were separate
- Usually makes sense
- Previous examples were all superadditive
- Given this, always efficient for grand coalition to form

Convexity

- A game is convex if for all coalitions A, B, v(AUB)-v(B) ≥ v(A)-v(A∩B) (i.e., v is supermodular)
- One interpretation: the marginal contribution of an agent is increasing in the size of the set that it is added to
- Previous examples were not convex (why?)
- In convex games, core is always nonempty
- One easy-to-compute solution in the core: agent i gets u(i) = v({1, 2, …, i}) - v({1, 2, …, i-1})
  - Marginal contribution scheme
  - Works for any ordering of the agents

The Shapley value [Shapley 1953]

- The marginal contribution scheme is unfair because it depends on the ordering of the agents
- One way to make it fair: average over all possible orderings
- Let MC(i, π) be the marginal contribution of i in ordering π
- Then i's Shapley value is Σ_{π}MC(i, π)/(n!)
- Always in the core for convex games
- … but not in general, even when core is nonempty, e.g.
  - v({1, 2, 3}) = v({1, 2}) = v({1, 3}) = 1,
  - v = 0 everywhere else

Axiomatic characterization of the Shapley value

- The Shapley value is the unique solution concept that satisfies:
  - Efficiency: the total utility is the value of the grand coalition, $\Sigma_{i \text{ in } N} u(i) = v(N)$
  - Symmetry: two symmetric players must receive the same utility
  - Dummy: if v(SU{i}) = v(S) for all S, then i must get 0
  - Additivity: if we add two games defined by v and w by letting (v+w)(S) = v(S) + w(S), then the utility for an agent in v+w should be the sum of her utilities in v and w
    - most controversial axiom

Computing a solution in the core

- Can use linear programming:
  - Variables: u(i)
  - Distribution constraint: $\Sigma_{i \text{ in } N} u(i) = v(N)$
  - Non-blocking constraints: for every S, $\Sigma_{i \text{ in } S} u(i) \geq v(S)$
- Problem: number of constraints exponential in number of players
- … but if the input explicitly specifies the value of every coalition, polynomial in input size
- … but is this practical?

A concise representation based on synergies [Conitzer & Sandholm AIJ06]

- Assume superadditivity
- Say that a coalition S is synergetic if there do not exist A, B with $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = S$, $v(S) = v(A) + v(B)$
- Value of non-synergetic coalitions can be derived from values of smaller coalitions
- So, only specify values for synergetic coalitions in the input

A useful lemma

- Lemma: For a given outcome, if there is a blocking coalition S (i.e., $\Sigma_{i \text{ in } S} u(i) < v(S)$), then there is also a synergetic blocking coalition
- Proof:
  - WLOG, suppose S is the smallest blocking coalition
  - Suppose S is not synergetic
  - So, there exist A, B with $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = S$, $v(S) = v(A) + v(B)$
  - $\Sigma_{i \text{ in } A} u(i) + \Sigma_{i \text{ in } B} u(i) = \Sigma_{i \text{ in } S} u(i) < v(S) = v(A) + v(B)$
  - Hence either $\Sigma_{i \text{ in } A} u(i) < v(A)$ or $\Sigma_{i \text{ in } B} u(i) < v(B)$
  - I.e., either A or B must be blocking
  - Contradiction!

Computing a solution in the core under synergy representation

- Can again use linear programming:
  - Variables: u(i)
  - Distribution constraint: $\Sigma_{i \text{ in } N}$u(i) = v(N)
  - Non-blocking constraints: for every synergetic S, $\Sigma_{i \text{ in } S}$u(i) ≥ v(S)
- Still requires us to know v(N)
- If we do not know this, computing a solution in the core is NP-hard
- This is because computing v(N) is NP-hard
- So, the hard part is not the strategic constraints, but computing what the grand coalition can do
- If the game is convex, then a solution in the core can be constructed in polynomial time even without knowing v(N)

Other concise representations of coalitional games

- [Deng & Papadimitriou 94]: agents are vertices of a graph, edges have weights, value of coalition = sum of weights of edges in coalition
- [Conitzer & Sandholm 04]: represent game as sum of smaller games (each of which involves only a few agents)
- [Ieong & Shoham 05]: multiple rules of the form (1 and 3 and (not 4) → 7), value of coalition = sum of values of rules that apply to it
  - E.g., the above rule applies to coalition {1, 2, 3} (so it gets 7 from this rule), but not to {1, 3, 4} or {1, 2, 5} (so they get nothing from this rule)
  - Generalizes the above two representations (but not synergy-based representation)

Nucleolus [Schmeidler 1969]

- Always gives a solution in the core if there exists one
- Always uniquely determined
- A coalition's excess e(S) is v(S) - $\Sigma_{i \text{ in } S}$u(i)
- For a given outcome, list all coalitions' excesses in decreasing order
- E.g., consider
  - v({1, 2, 3}) = 6,
  - v({1, 2}) = v({1, 3}) = v({2, 3}) = 6,
  - v({1}) = v({2}) = v({3}) = 0
- For outcome (2, 2, 2), the list of excesses is 2, 2, 2, 0, -2, -2, -2 (coalitions of size 2, 3, 1, respectively)
- For outcome (3, 3, 0), the list of excesses is 3, 3, 0, 0, 0, -3, -3 (coalitions {1, 3}, {2, 3}; {1, 2}, {1, 2, 3}, {3}; {1}, {2})
- Nucleolus is the (unique) outcome that lexicographically minimizes the list of excesses
  - Lexicographic minimization = minimize the first entry first, then (fixing the first entry) minimize the second one, etc.

Marriage contract problem

[Babylonian Talmud, 0-500AD]

- A man has three wives
- Their marriage contracts specify that they should, respectively, receive 100, 200, and 300 in case of his death
- … but there may not be that much money to go around…
- Talmud recommends:
  - If 100 is available, each agent (wife) gets 33 1/3
  - If 200 is available, agent 1 gets 50, other two get 75 each
  - If 300 is available, agent 1 gets 50, agent 2 gets 100, agent 3 gets 150
- ?
- Define $v(S) = \max\{0, \text{money available} - \Sigma_{i \text{ in } N-S} \text{claim}(i)\}$
  - Any coalition can walk away and obtain 0
  - Any coalition can pay off agents outside the coalition and divide the remainder
- Talmud recommends the nucleolus! [Aumann & Maschler 85]