CPS 296.1
Voting and social choice

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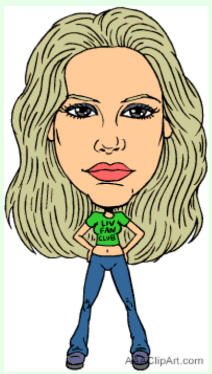
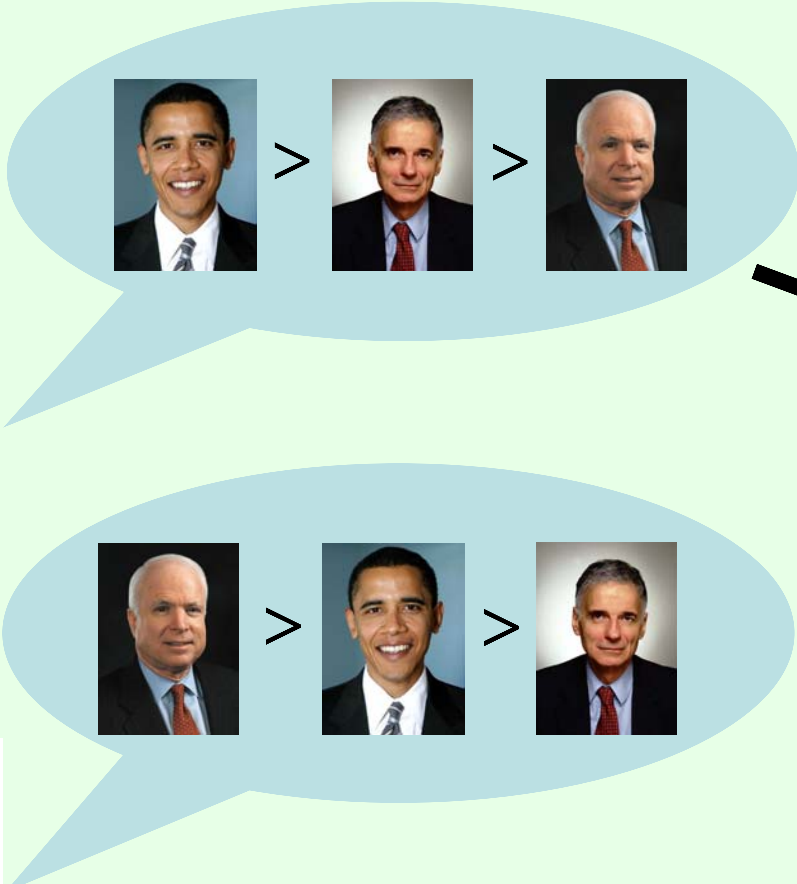
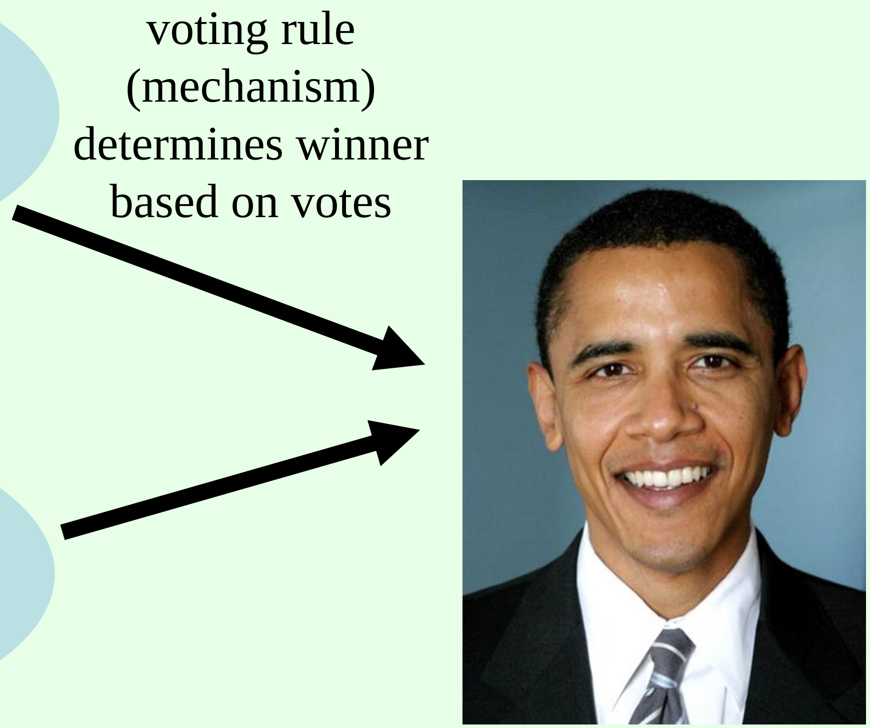
Voting over alternatives







- Can vote over other things too
  - Where to go for dinner tonight, other joint plans, …

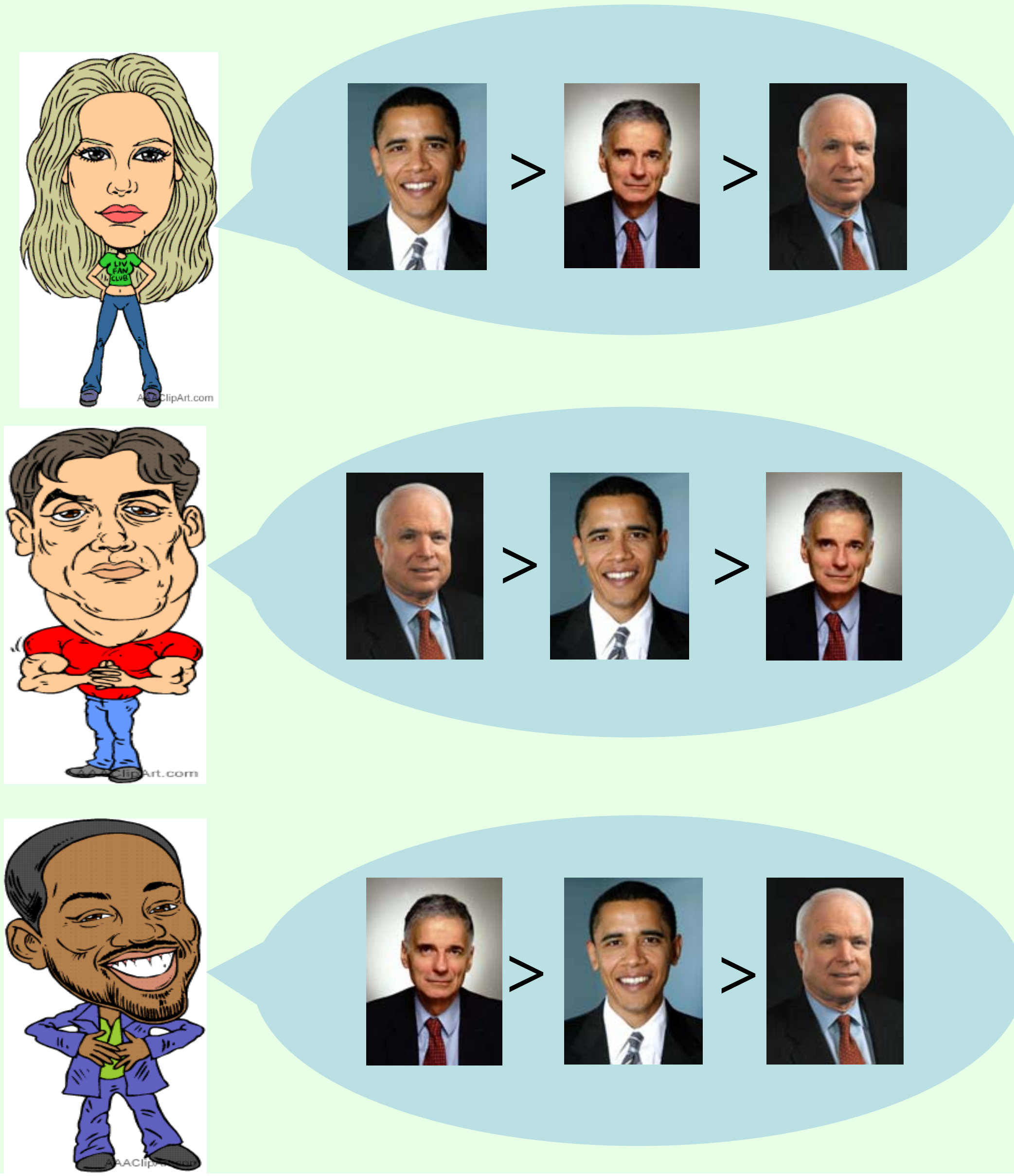
Voting (rank aggregation)

- Set of m candidates (aka. alternatives, outcomes)
- n voters; each voter ranks all the candidates
  - E.g., for set of candidates {a, b, c, d}, one possible vote is b > a > d > c
  - Submitted ranking is called a vote
- A voting rule takes as input a vector of votes (submitted by the voters), and as output produces either:
  - the winning candidate, or
  - an aggregate ranking of all candidates
- Can vote over just about anything
  - political representatives, award nominees, where to go for dinner tonight, joint plans, allocations of tasks/resources, …
  - Also can consider other applications: e.g., aggregating search engine’s rankings into a single ranking

Example voting rules

- Scoring rules are defined by a vector $(a_1, a_2, \ldots, a_m)$; being ranked ith in a vote gives the candidate a_i points
  - Plurality is defined by (1, 0, 0, …, 0) (winner is candidate that is ranked first most often)
  - Veto (or anti-plurality) is defined by (1, 1, …, 1, 0) (winner is candidate that is ranked last the least often)
  - Borda is defined by (m-1, m-2, …, 0)
- Plurality with (2-candidate) runoff: top two candidates in terms of plurality score proceed to runoff; whichever is ranked higher than the other by more voters, wins
- Single Transferable Vote (STV, aka. Instant Runoff): candidate with lowest plurality score drops out; if you voted for that candidate, your vote transfers to the next (live) candidate on your list; repeat until one candidate remains
- Similar runoffs can be defined for rules other than plurality

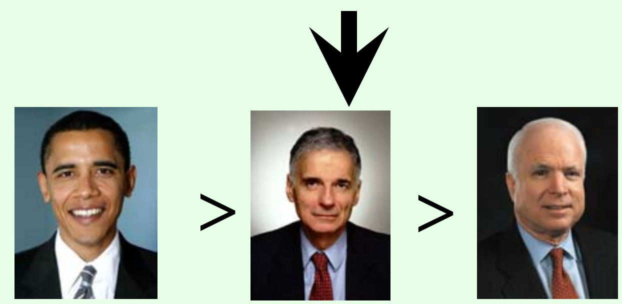
Pairwise elections



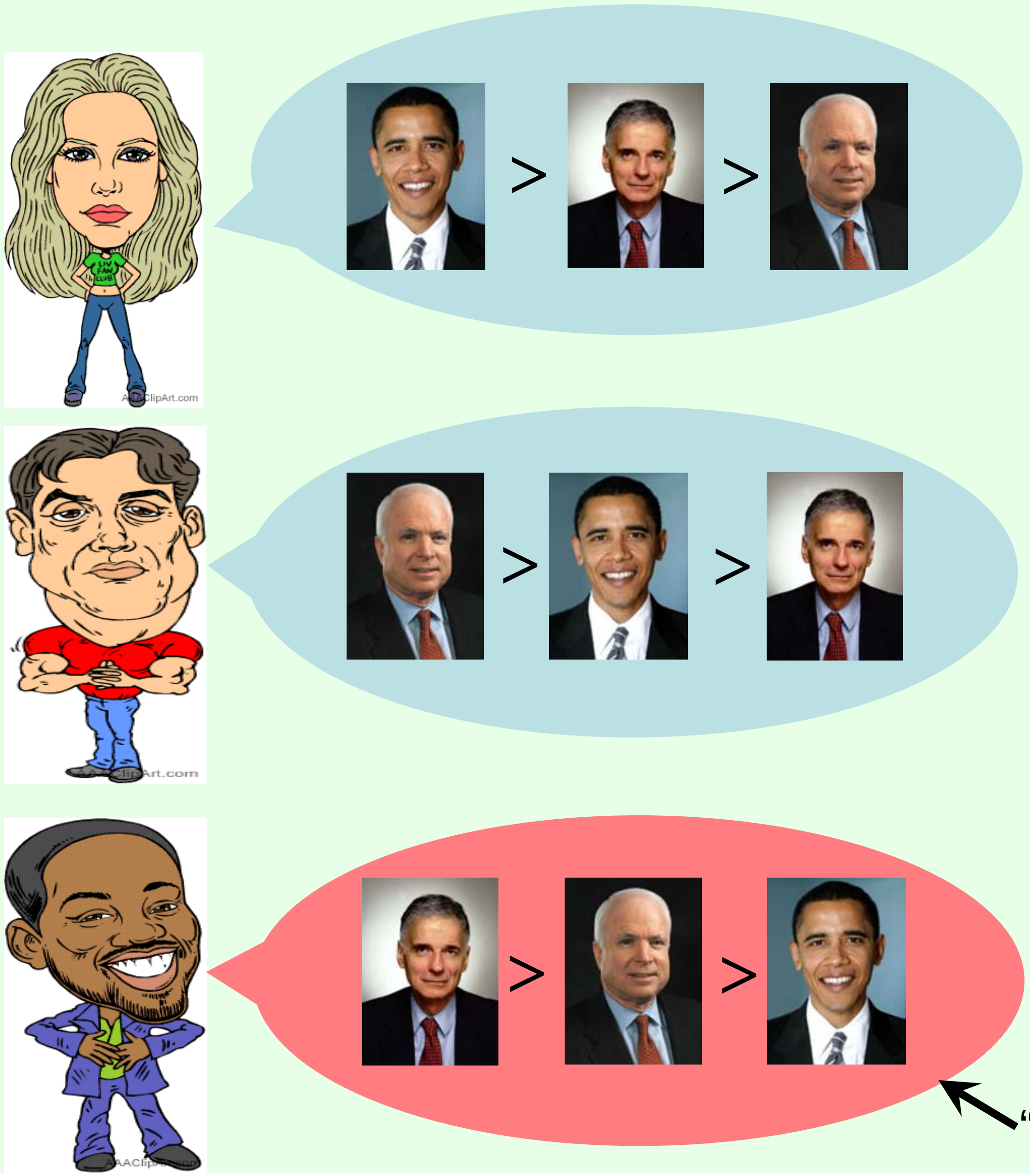
two votes prefer Obama to McCain

two votes prefer Obama to Nader

two votes prefer Nader to McCain

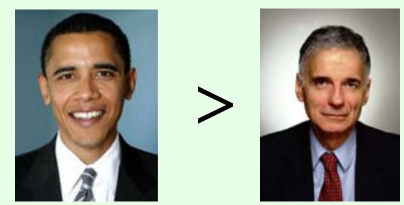


Condorcet cycles



two votes prefer McCain to Obama

two votes prefer Obama to Nader



two votes prefer Nader to McCain

?

“weird” preferences

Voting rules based on pairwise elections

- Copeland: candidate gets two points for each pairwise election it wins, one point for each pairwise election it ties
- Maximin (aka. Simpson): candidate whose worst pairwise result is the best wins
- Slater: create an overall ranking of the candidates that is inconsistent with as few pairwise elections as possible
  - NP-hard!
- Cup/pairwise elimination: pair candidates, losers of pairwise elections drop out, repeat

Even more voting rules…

- Kemeny: create an overall ranking of the candidates that has as few *disagreements* as possible (where a disagreement is with a vote on a pair of candidates)
  - NP-hard!
- Bucklin: start with k=1 and increase k gradually until some candidate is among the top k candidates in more than half the votes; that candidate wins
- Approval (not a ranking-based rule): every voter labels each candidate as approved or disapproved, candidate with the most approvals wins
- … how do we choose a rule from all of these rules?
- How do we know that there does not exist another, "perfect" rule?
- Let us look at some criteria that we would like our voting rule to satisfy

Condorcet criterion

- A candidate is the Condorcet winner if it wins all of its pairwise elections
- Does not always exist…
- … but the Condorcet criterion says that if it does exist, it should win

- Many rules do not satisfy this
- E.g. for plurality:
  - b > a > c > d
  - c > a > b > d
  - d > a > b > c
- a is the Condorcet winner, but it does not win under plurality

Majority criterion

- If a candidate is ranked first by most votes, that candidate should win
  - Relationship to Condorcet criterion?

- Some rules do not even satisfy this
- E.g. Borda:
  - a > b > c > d > e
  - a > b > c > d > e
  - c > b > d > e > a
- a is the majority winner, but it does not win under Borda

Monotonicity criteria

- Informally, monotonicity means that “ranking a candidate higher should help that candidate,” but there are multiple nonequivalent definitions
- A weak monotonicity requirement: if
  - candidate w wins for the current votes,
  - we then improve the position of w in some of the votes and leave everything else the same,

  then w should still win.
- E.g., STV does not satisfy this:
  - 7 votes b > c > a
  - 7 votes a > b > c
  - 6 votes c > a > b
- c drops out first, its votes transfer to a, a wins
- But if 2 votes b > c > a change to a > b > c, b drops out first, its 5 votes transfer to c, and c wins

Monotonicity criteria…

- A strong monotonicity requirement: if
  - candidate w wins for the current votes,
  - we then change the votes in such a way that for each vote, if a candidate c was ranked below w originally, c is still ranked below w in the new vote

  then w should still win.
- Note the other candidates can jump around in the vote, as long as they don't jump ahead of w
- None of our rules satisfy this

Independence of irrelevant alternatives

- Independence of irrelevant alternatives criterion: if
  - the rule ranks a above b for the current votes,
  - we then change the votes but do not change which is ahead between a and b in each vote

  then a should still be ranked ahead of b.
- None of our rules satisfy this

Arrow's impossibility theorem [1951]

- Suppose there are at least 3 candidates
- Then there exists no rule that is simultaneously:
  - Pareto efficient (if all votes rank a above b, then the rule ranks a above b),
  - nondictatorial (there does not exist a voter such that the rule simply always copies that voter's ranking), and
  - independent of irrelevant alternatives

Muller-Satterthwaite impossibility theorem [1977]

- Suppose there are at least 3 candidates
- Then there exists no rule that simultaneously:
  - satisfies unanimity (if all votes rank a first, then a should win),
  - is nondictatorial (there does not exist a voter such that the rule simply always selects that voter's first candidate as the winner), and
  - is monotone (in the strong sense).

Manipulability

- Sometimes, a voter is better off revealing her preferences insincerely, aka. manipulating
- E.g. plurality
  - Suppose a voter prefers a > b > c
  - Also suppose she knows that the other votes are
    - 2 times b > c > a
    - 2 times c > a > b
  - Voting truthfully will lead to a tie between b and c
  - She would be better off voting e.g. b > a > c, guaranteeing b wins

- All our rules are (sometimes) manipulable

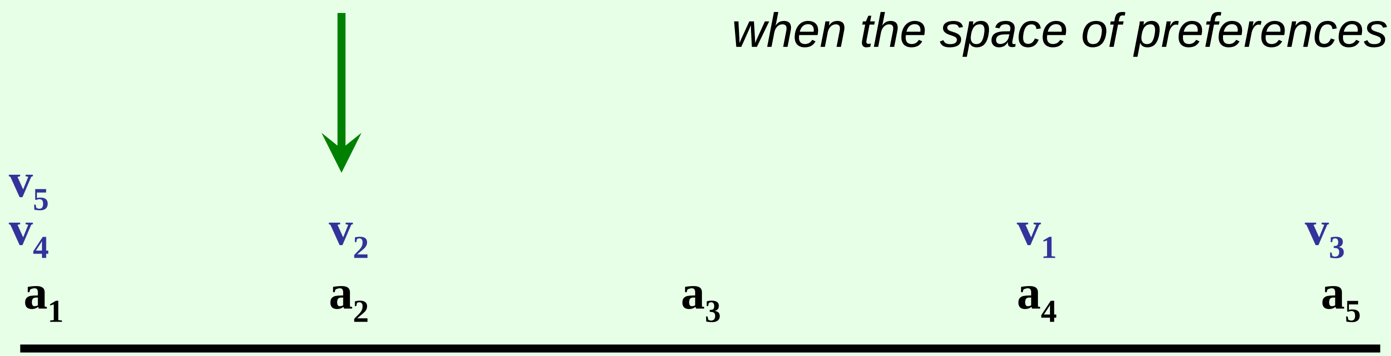
Gibbard-Satterthwaite impossibility theorem

- Suppose there are at least 3 candidates
- There exists no rule that is simultaneously:
  - onto (for every candidate, there are some votes that would make that candidate win),
  - nondictatorial, and
  - nonmanipulable

Single-peaked preferences

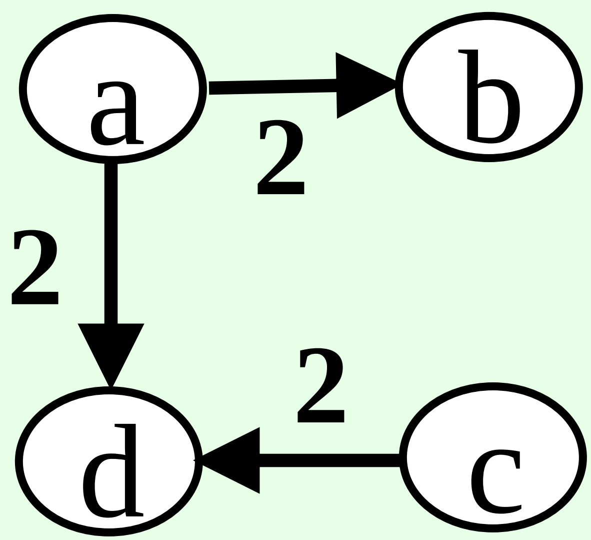
- Suppose candidates are ordered on a line
- Every voter prefers candidates that are closer to her most preferred candidate
- Let every voter report only her most preferred candidate ("peak")
- Choose the median voter's peak as the winner
  - This will also be the Condorcet winner
- Nonmanipulable!

Impossibility results do not necessarily hold when the space of preferences is restricted



Pairwise election graphs

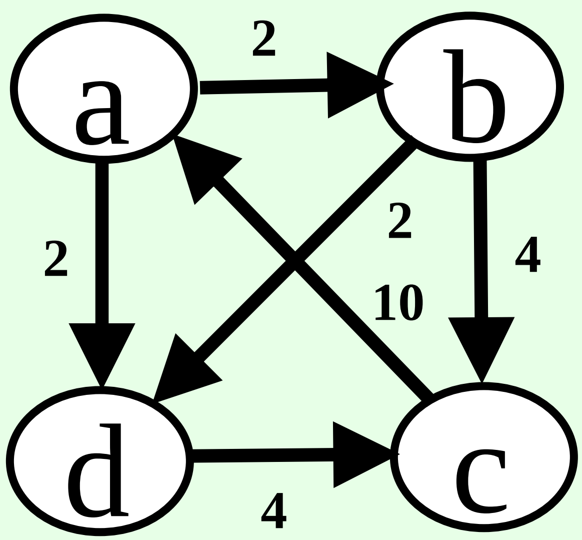
- Pairwise election between *a* and *b*: compare how often *a* is ranked above *b* vs. how often *b* is ranked above *a*
- Graph representation: edge from winner to loser (no edge if tie), weight = margin of victory
- E.g., for votes *a* > *b* > *c* > *d*, *c* > *a* > *d* > *b* this gives



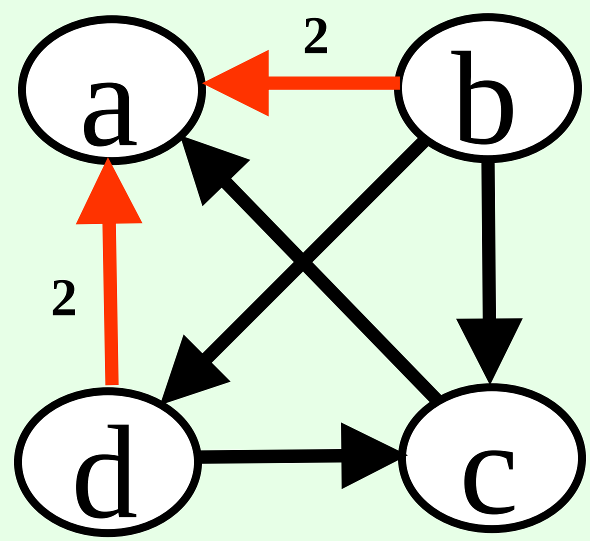
Kemeny on pairwise election graphs

- Final ranking = acyclic tournament graph
  - Edge (a, b) means a ranked above b
  - Acyclic = no cycles, tournament = edge between every pair
- Kemeny ranking seeks to minimize the total weight of the inverted edges

pairwise election graph



Kemeny ranking

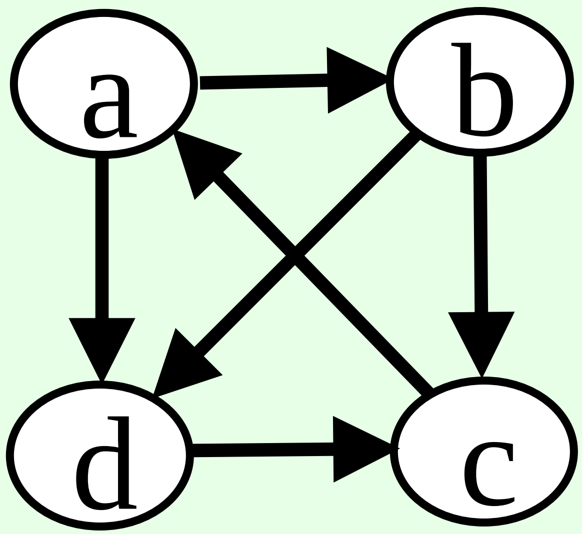


(b > d > c > a)

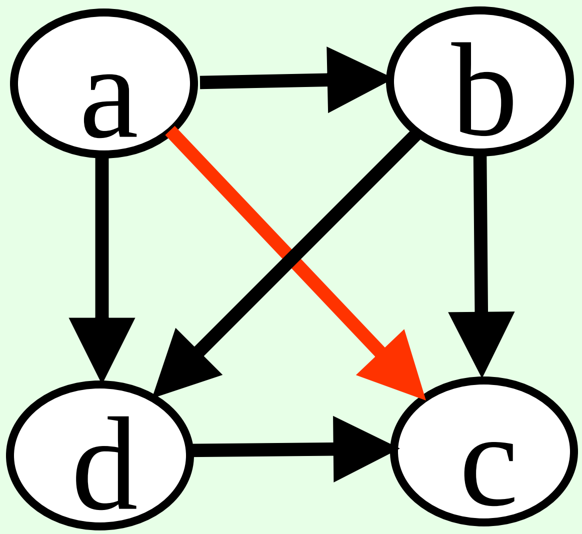
Slater on pairwise election graphs

- Final ranking = acyclic tournament graph
- Slater ranking seeks to minimize the number of inverted edges

pairwise election graph



Slater ranking



(a > b > d > c)

An integer program for computing Kemeny/Slater rankings

$y_{(a, b)}$ is 1 if a is ranked below b, 0 otherwise

$w_{(a, b)}$ is the weight on edge (a, b) (if it exists)

in the case of Slater, weights are always 1

minimize: $\Sigma_{e \in E} w_e y_e$

subject to:

for all $a, b \in V$, $y_{(a, b)} + y_{(b, a)} = 1$

for all $a, b, c \in V$, $y_{(a, b)} + y_{(b, c)} + y_{(c, a)} \geq 1$

Some computational issues in social choice

- Sometimes computing the winner/aggregate ranking is hard
  – E.g. for Kemeny and Slater rules this is NP-hard
- For some rules (e.g., STV), computing a successful manipulation is NP-hard
  – Manipulation being hard is a **good** thing (circumventing Gibbard-Satterthwaite?)… But would like something stronger than NP-hardness
  – Also: work on the complexity of controlling the outcome of an election by influencing the list of candidates/schedule of the Cup rule/etc.
- Preference elicitation:
  – We may not want to force each voter to rank **all** candidates;
  – Rather, we can selectively query voters for parts of their ranking, according to some algorithm, to obtain a good aggregate outcome
- Combinatorial alternative spaces:
  – Suppose there are multiple interrelated issues that each need a decision
  – Exponentially sized alternative spaces
- Different models such as ranking webpages (pages “vote” on each other by linking)