

CPS 570: Artificial Intelligence

Markov processes and Hidden Markov Models (HMMs)

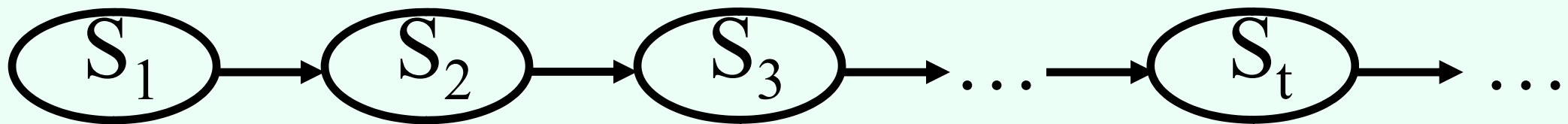
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Motivation

- The Bayes nets we considered so far were **static**: they referred to a single point in time
 - E.g., medical diagnosis
- Agent needs to model how the world **evolves**
 - Speech recognition software needs to process speech over time
 - Artificially intelligent software assistant needs to keep track of user's intentions over time
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Markov processes

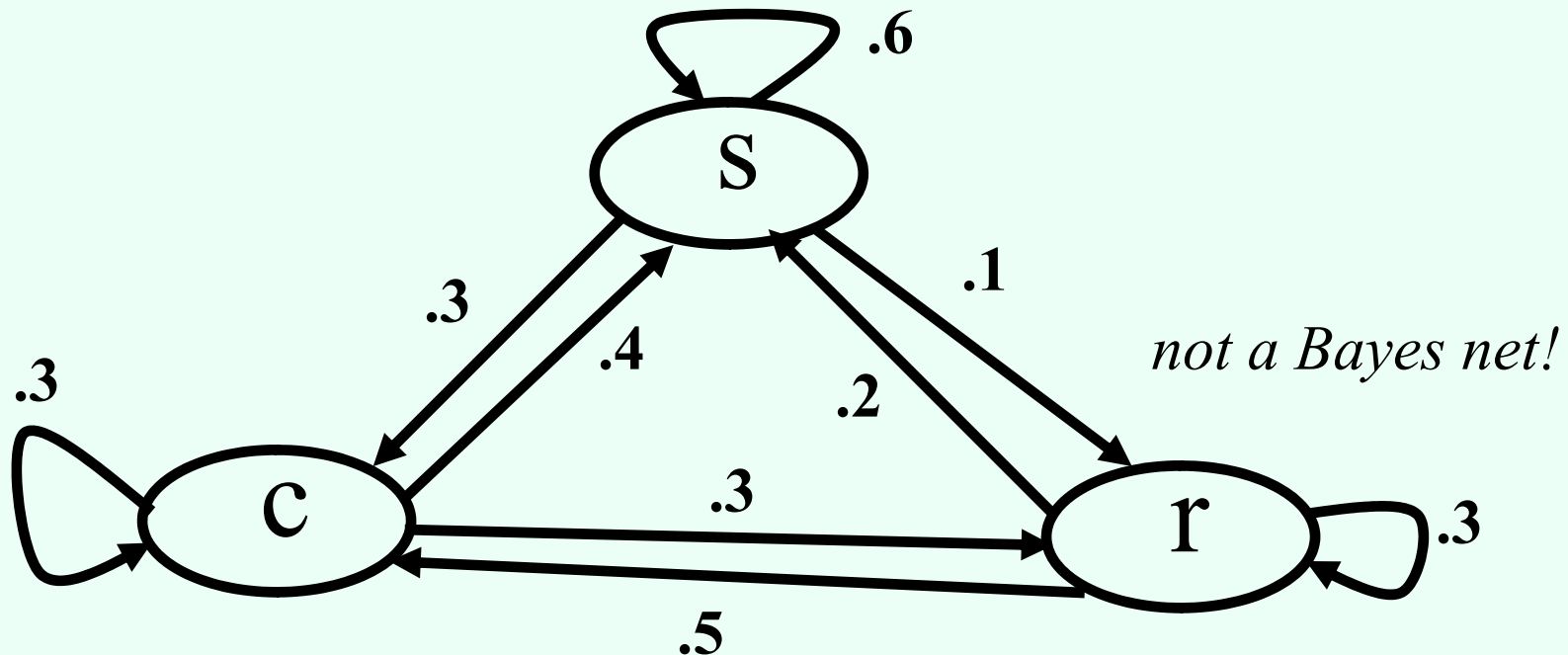
- We have time periods $t = 0, 1, 2, \dots$
- In each period t , the world is in a certain state S_t
- The **Markov assumption**: given S_t , S_{t+1} is independent of all S_i with $i < t$
 - $P(S_{t+1} | S_1, S_2, \dots, S_t) = P(S_{t+1} | S_t)$
 - Given the current state, history tells us nothing more about the future



- Typically, all the CPTs are the same:
- For all t , $P(S_{t+1} = j | S_t = i) = a_{ij}$ (**stationarity assumption**)

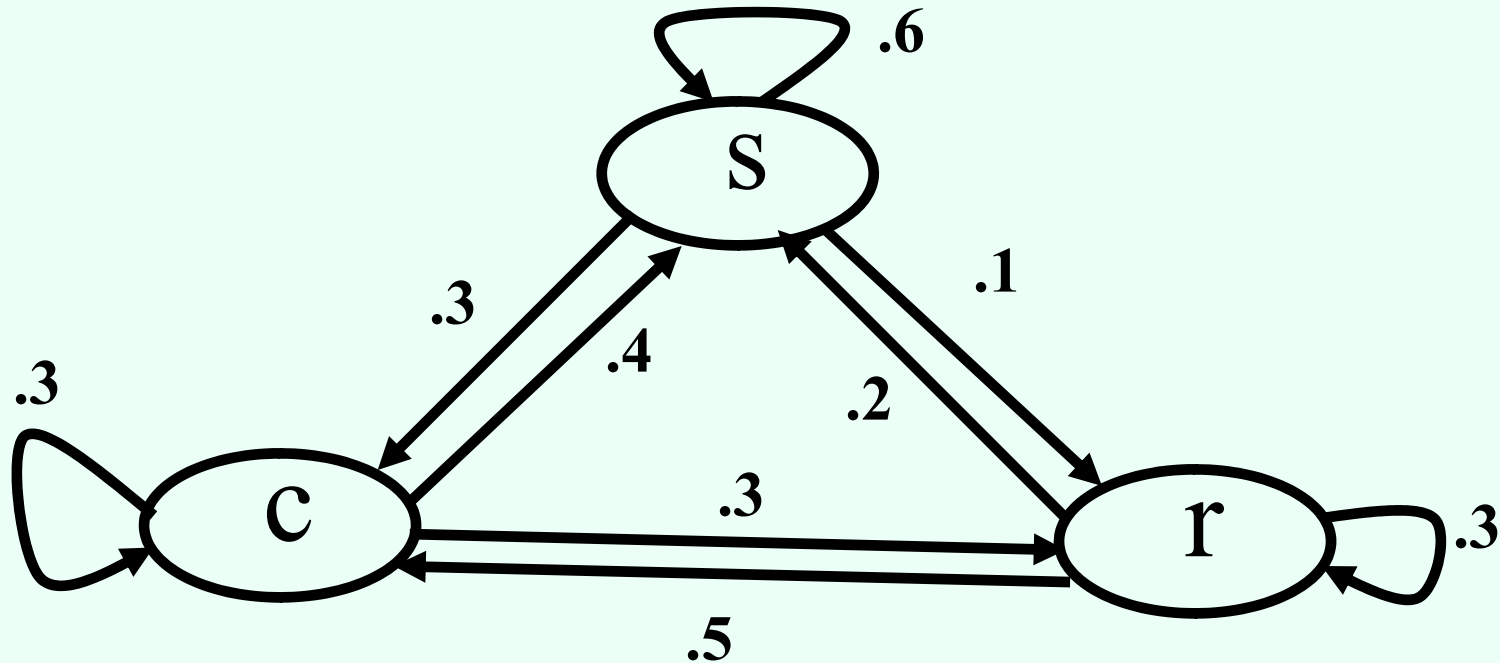
Weather example

- S_t is one of $\{s, c, r\}$ (sun, cloudy, rain)
- Transition probabilities:



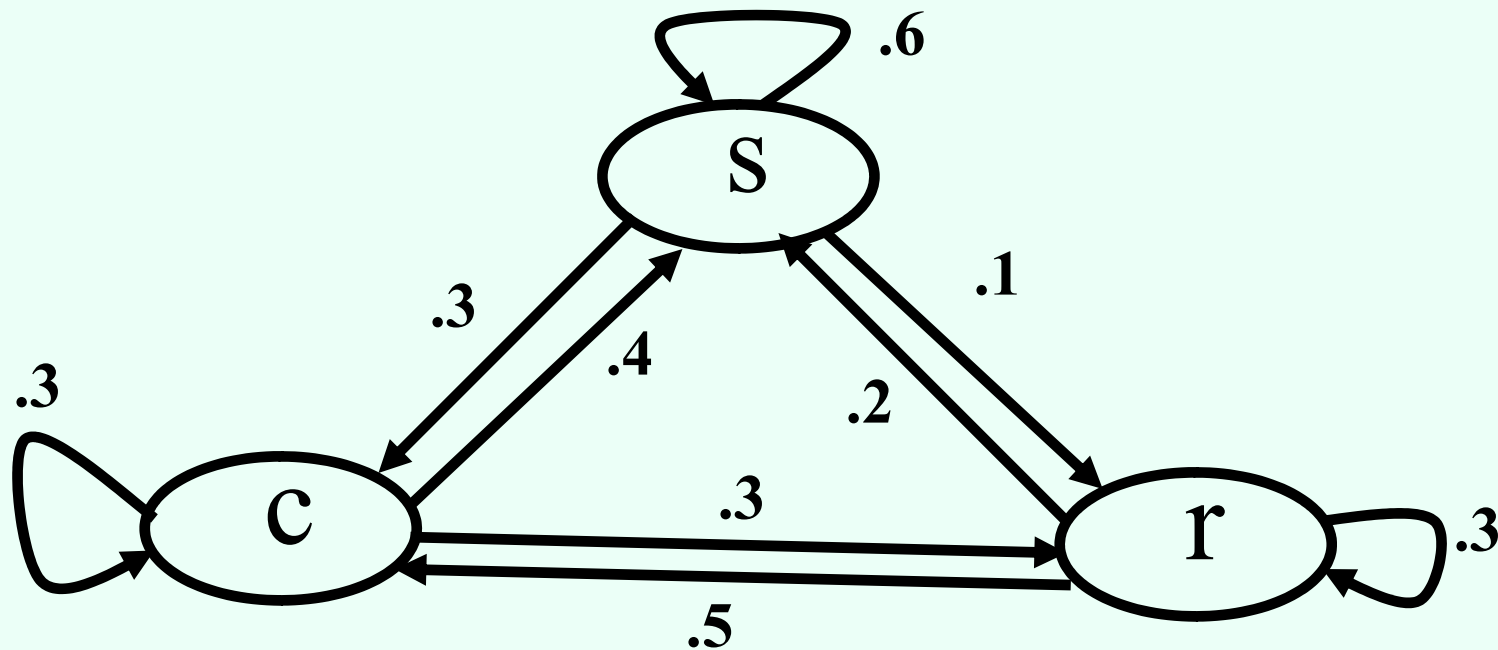
- Also need to specify an initial distribution $P(S_0)$
- Throughout, assume $P(S_0 = s) = 1$

Weather example...



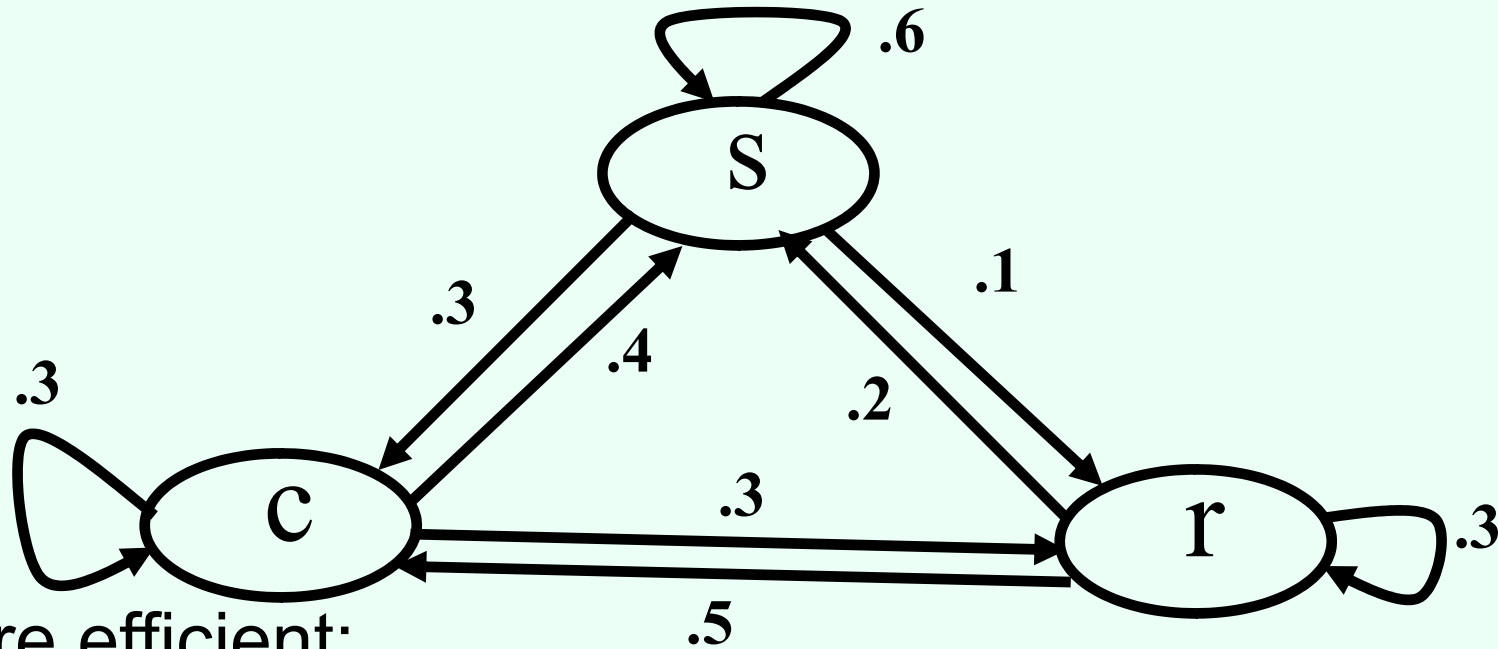
- What is the probability that it rains two days from now? $P(S_2 = r)$
- $P(S_2 = r) = P(S_2 = r, S_1 = r) + P(S_2 = r, S_1 = s) + P(S_2 = r, S_1 = c) = .1 * .3 + .6 * .1 + .3 * .3 = .18$

Weather example...



- What is the probability that it rains **three** days from now?
- Computationally inefficient way: $P(S_3 = r) = P(S_3 = r, S_2 = r, S_1 = r) + P(S_3 = r, S_2 = r, S_1 = s) + \dots$
- For n periods into the future, need to sum over 3^{n-1} paths

Weather example...



- More efficient:
- $P(S_3 = r) = P(S_3 = r, S_2 = r) + P(S_3 = r, S_2 = s) + P(S_3 = r, S_2 = c) = P(S_3 = r | S_2 = r)P(S_2 = r) + P(S_3 = r | S_2 = s)P(S_2 = s) + P(S_3 = r | S_2 = c)P(S_2 = c)$
- Only hard part: figure out $P(S_2)$
- Main idea: compute distribution $P(S_1)$, then $P(S_2)$, then $P(S_3)$
- Linear in number of periods!

example on board

Stationary distributions

- As t goes to infinity, “generally,” the distribution $P(S_t)$ will converge to a **stationary** distribution
- A distribution given by probabilities π_i (where i is a state) is stationary if:

$$P(S_t = i) = \pi_i \text{ means that } P(S_{t+1} = i) = \pi_i$$

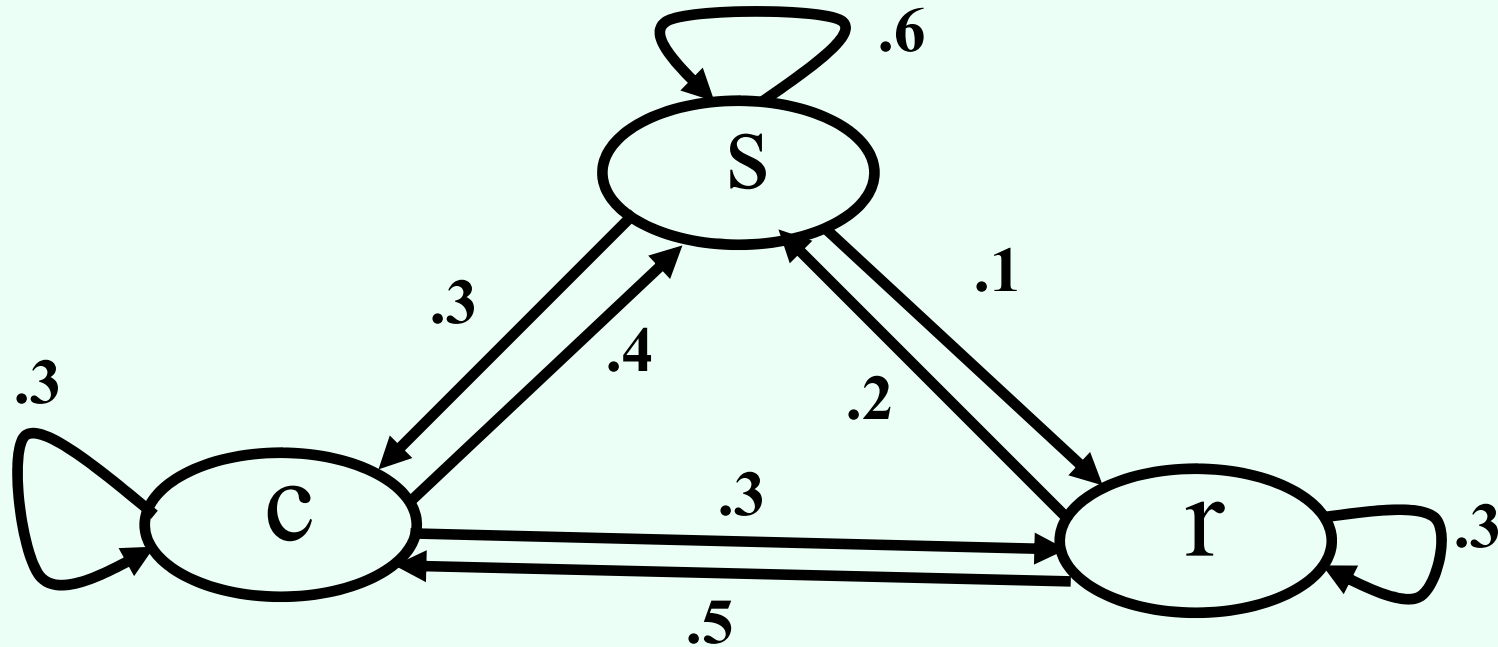
- Of course,

$$P(S_{t+1} = i) = \sum_j P(S_{t+1} = i, S_t = j) = \sum_j P(S_t = j) a_{ji}$$

- So, stationary distribution is defined by

$$\pi_i = \sum_j \pi_j a_{ji}$$

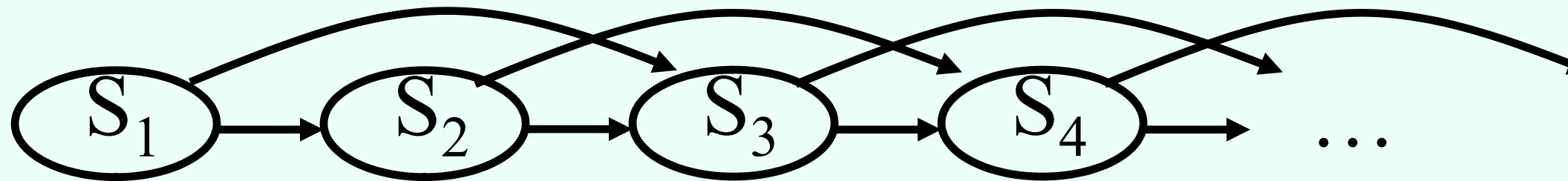
Computing the stationary distribution



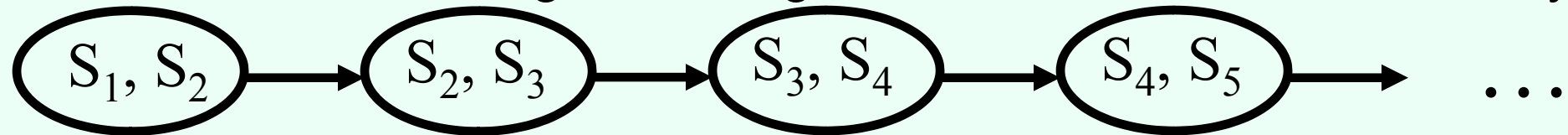
- $\pi_s = .6\pi_s + .4\pi_c + .2\pi_r$
- $\pi_c = .3\pi_s + .3\pi_c + .5\pi_r$
- $\pi_r = .1\pi_s + .3\pi_c + .3\pi_r$

Restrictiveness of Markov models

- Are past and future really independent given current state?
- E.g., suppose that when it rains, it rains for at most 2 days



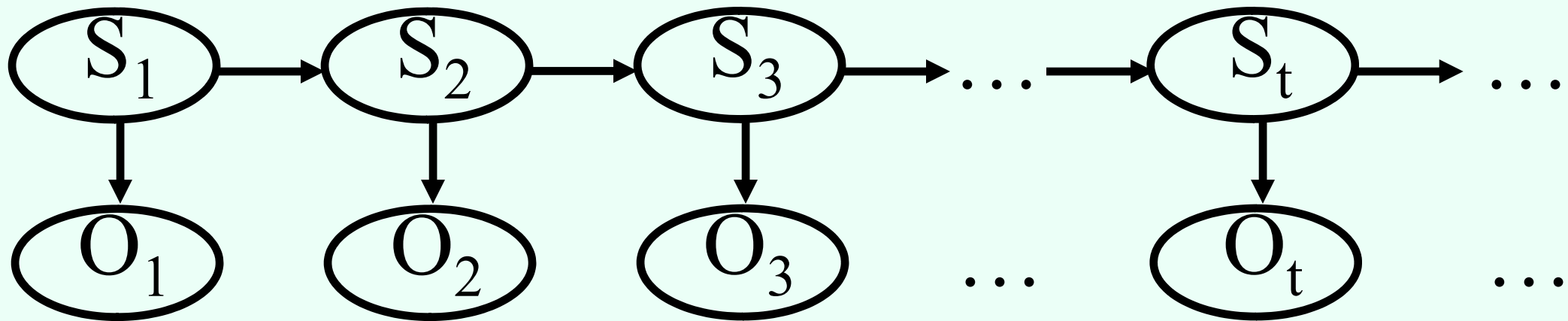
- **Second-order** Markov process
- Workaround: change meaning of “state” to events of last 2 days



- Another approach: add more information to the state
- E.g., the full state of the world would include whether the sky is full of water
 - Additional information may not be observable
 - Blowup of number of states...

Hidden Markov models (HMMs)

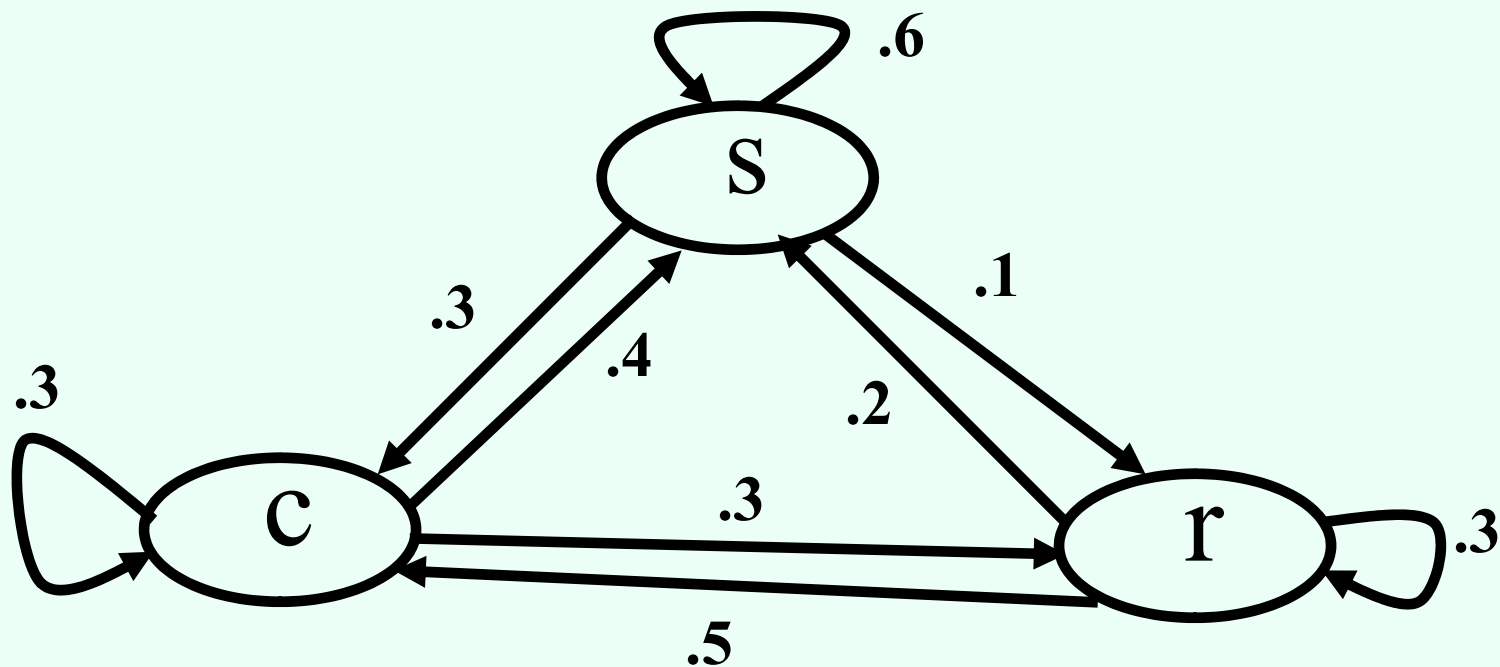
- Same as Markov model, except we cannot see the state
- Instead, we only see an **observation** each period, which depends on the current state



- Still need a transition model: $P(S_{t+1} = j \mid S_t = i) = a_{ij}$
- Also need an observation model: $P(O_t = k \mid S_t = i) = b_{ik}$

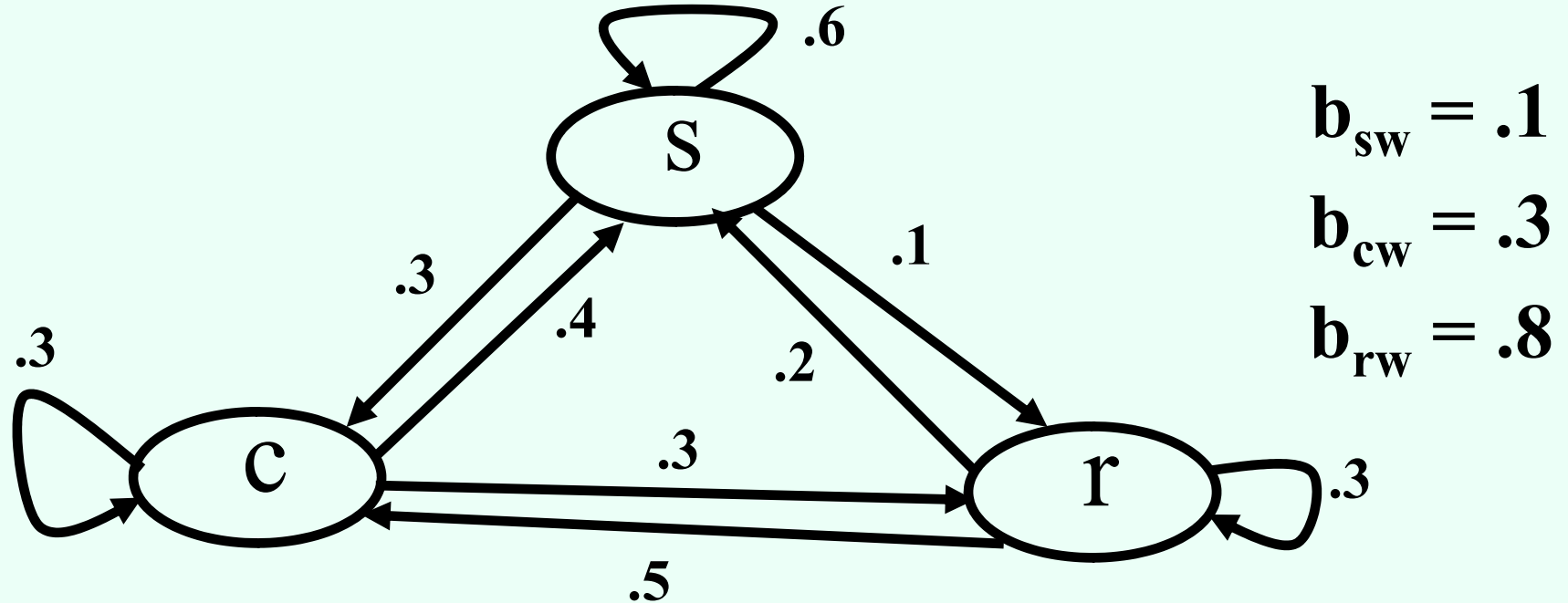
Weather example extended to HMM

- Transition probabilities:



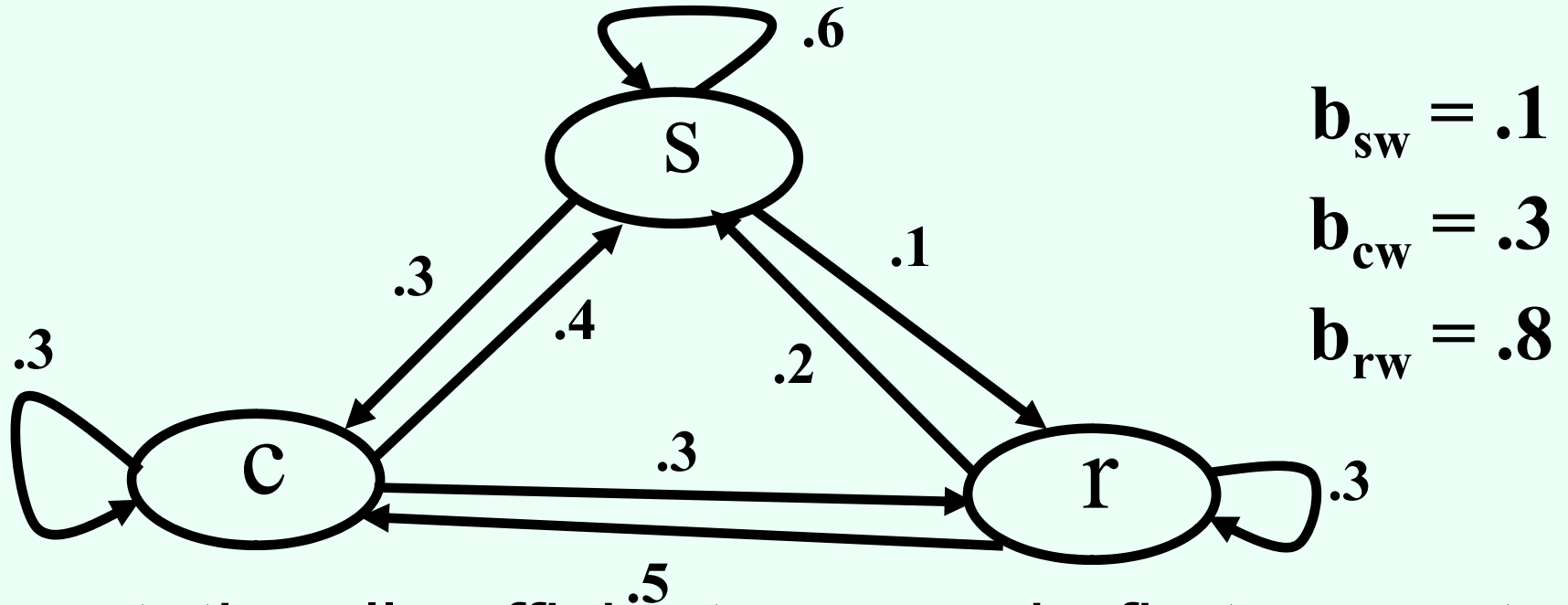
- Observation: labmate wet or dry
- $b_{sw} = .1$, $b_{cw} = .3$, $b_{rw} = .8$

HMM weather example: a question



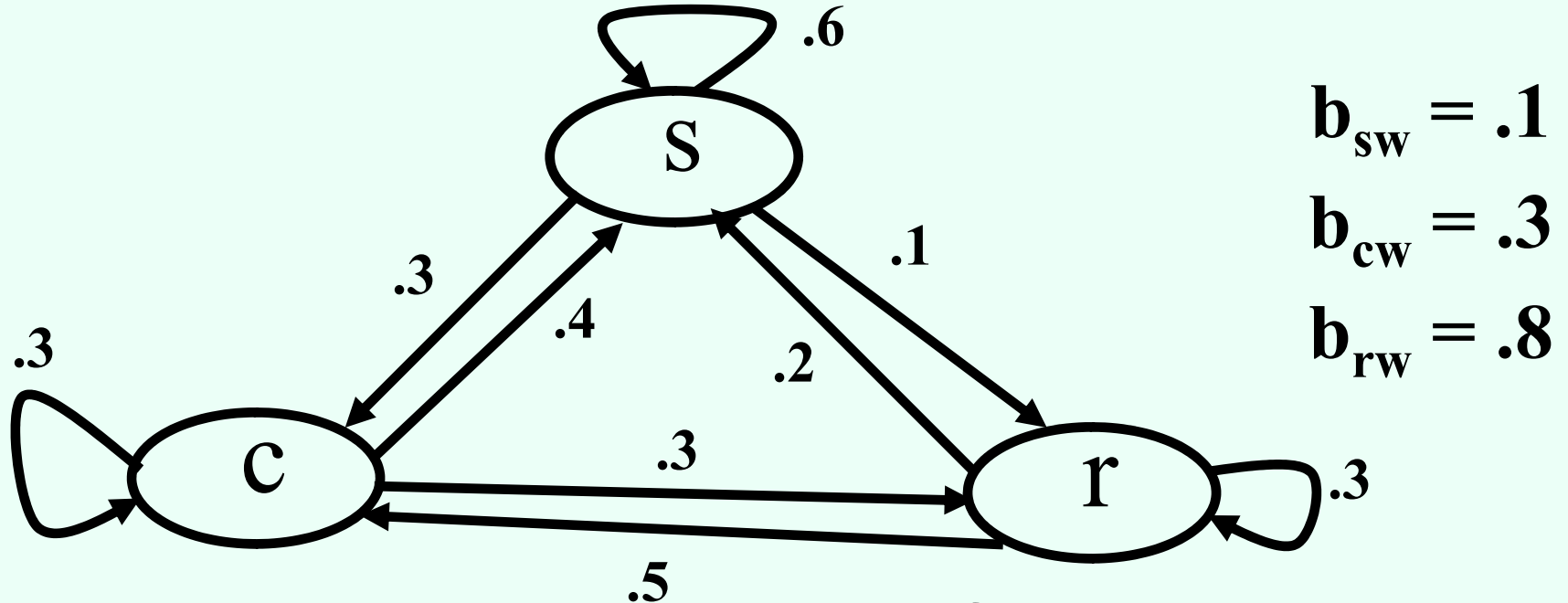
- You have been stuck in the lab for three days (!)
- On those days, your labmate was dry, wet, wet, respectively
- What is the probability that it is now raining outside?
- $P(S_2 = r \mid O_0 = d, O_1 = w, O_2 = w)$
- By Bayes' rule, really want to know $P(S_2, O_0 = d, O_1 = w, O_2 = w)$

Solving the question



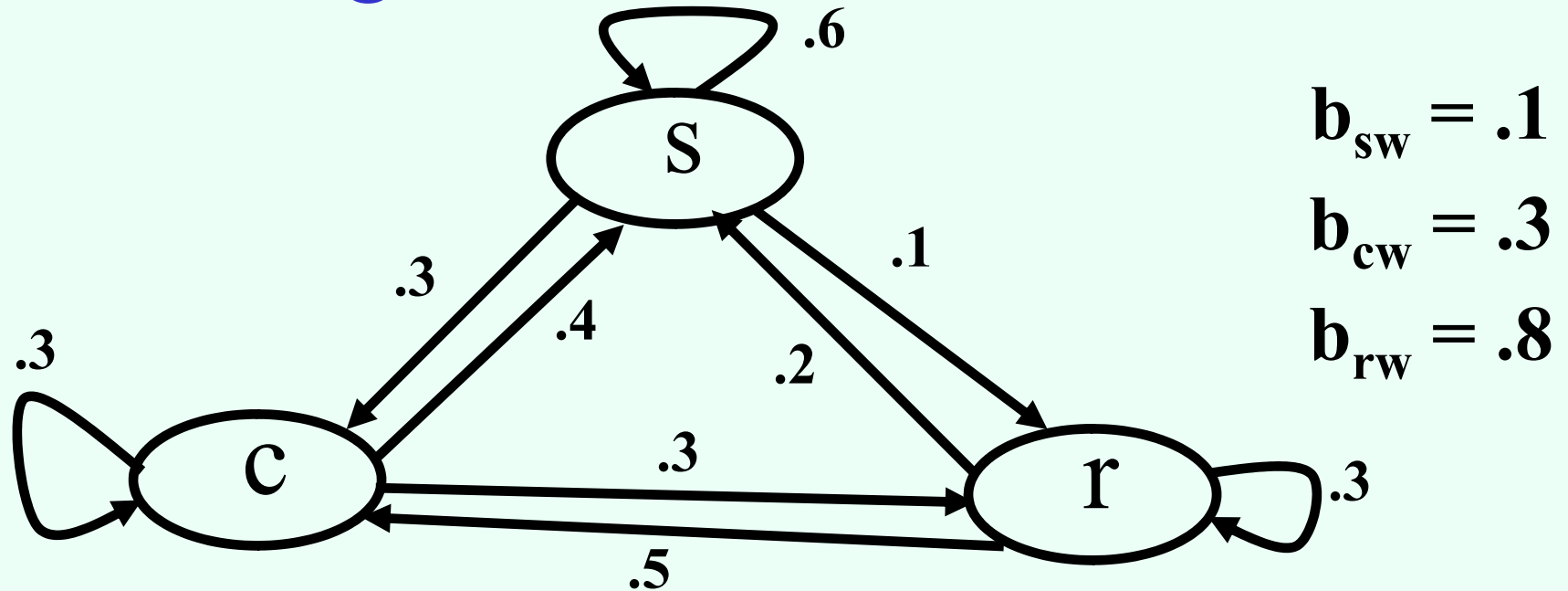
- Computationally efficient approach: first compute $P(S_1 = i, O_0 = d, O_1 = w)$ for all states i
- General case: solve for $P(S_t, O_0 = o_0, O_1 = o_1, \dots, O_t = o_t)$ for $t=1$, then $t=2, \dots$. This is called **monitoring**
- $P(S_t, O_0 = o_0, O_1 = o_1, \dots, O_t = o_t) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1}, O_0 = o_0, O_1 = o_1, \dots, O_{t-1} = o_{t-1}) P(S_t | S_{t-1} = s_{t-1}) P(O_t = o_t | S_t)$

Predicting further out



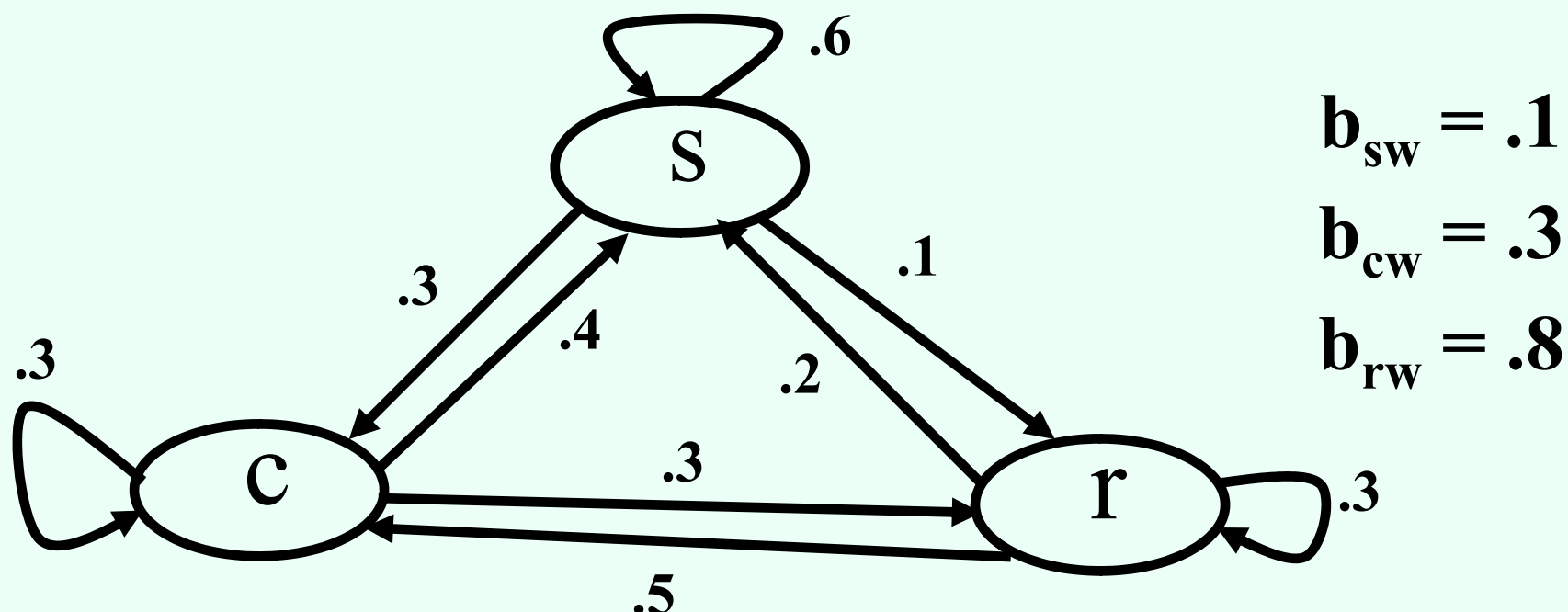
- You have been stuck in the lab for three days
- On those days, your labmate was dry, wet, wet, respectively
- What is the probability that **two days from now** it will be raining outside?
- $P(S_4 = r \mid O_0 = d, O_1 = w, O_2 = w)$

Predicting further out, continued...



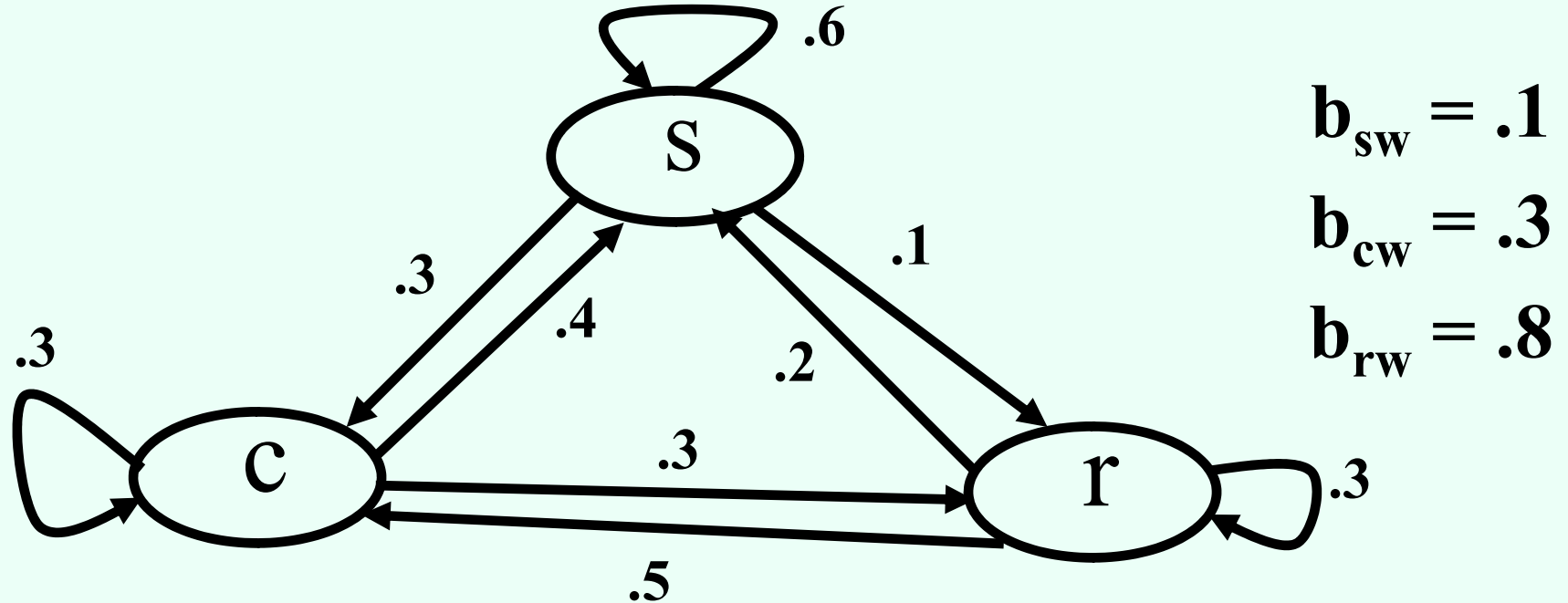
- Want to know: $P(S_4 = r \mid O_0 = d, O_1 = w, O_2 = w)$
- Already know how to get: $P(S_2 \mid O_0 = d, O_1 = w, O_2 = w)$
- $P(S_3 = r \mid O_0 = d, O_1 = w, O_2 = w) =$
 $\sum_{s_2} P(S_3 = r, S_2 = s_2 \mid O_0 = d, O_1 = w, O_2 = w)$
 $\sum_{s_2} P(S_3 = r \mid S_2 = s_2)P(S_2 = s_2 \mid O_0 = d, O_1 = w, O_2 = w)$
- Etc. for S_4
- So: monitoring first, then straightforward Markov process updates

Integrating newer information



- You have been stuck in the lab for **four** days (!)
- On those days, your labmate was dry, wet, wet, dry respectively
- What is the probability that **two days ago** it was raining outside? $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d)$
 - Smoothing or hindsight problem

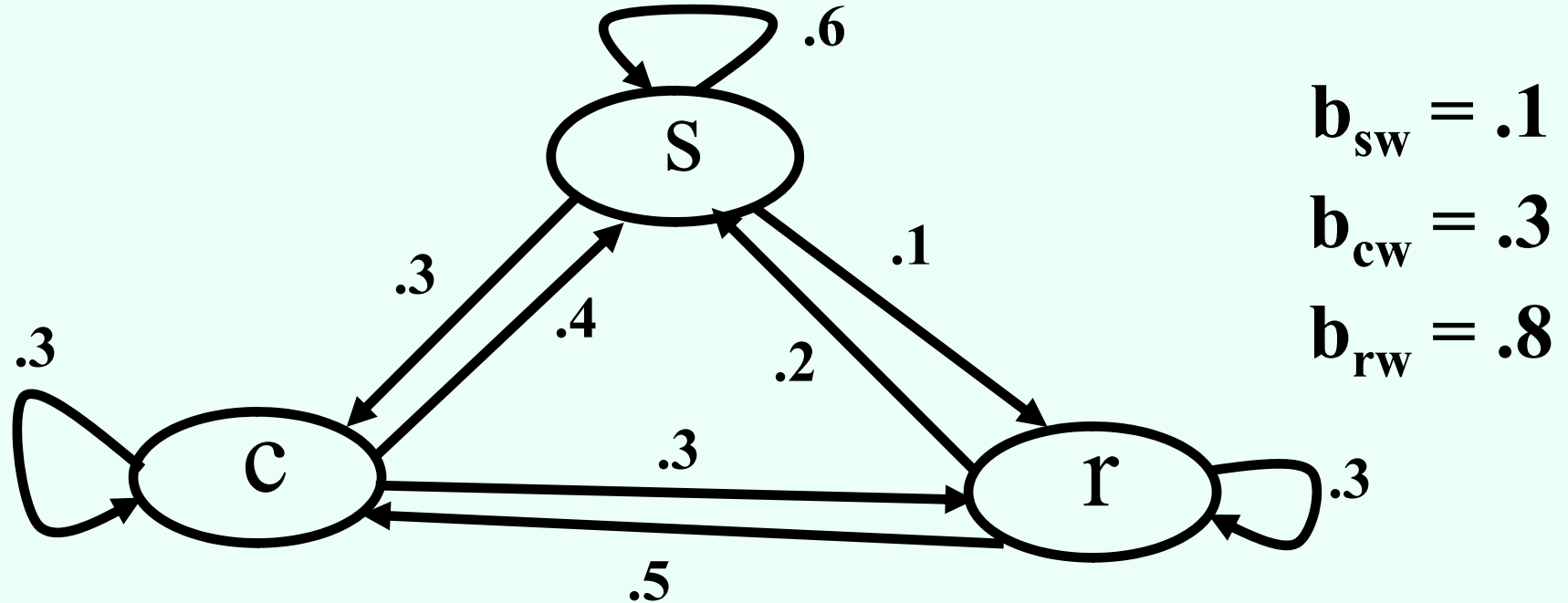
Hindsight problem continued...



- Want: $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d)$
- “Partial” application of Bayes’ rule:

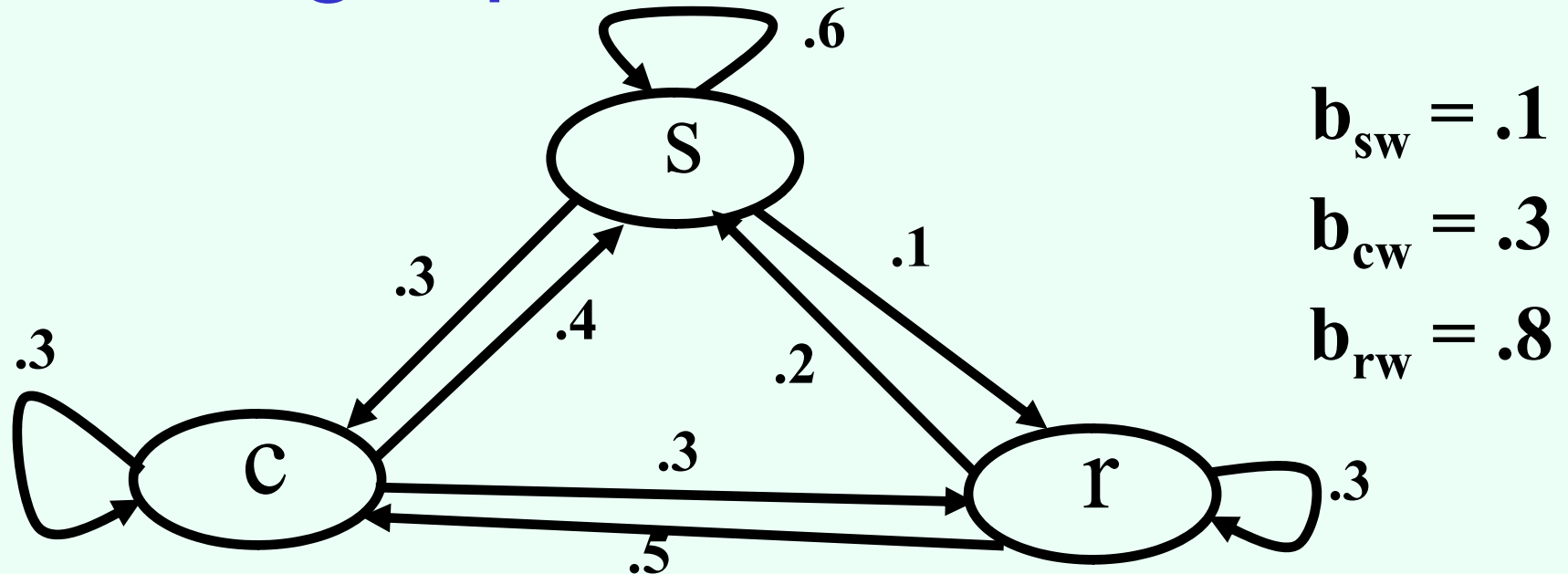
$$P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d) = \frac{P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)}{P(O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)}$$
- So really want to know $P(S_1, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$

Hindsight problem continued...



- Want to know $P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$
- $P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w) =$
 $P(S_1 = r \mid O_0 = d, O_1 = w) P(O_2 = w, O_3 = d \mid S_1 = r)$
- Already know how to compute $P(S_1 = r \mid O_0 = d, O_1 = w)$
- Just need to compute $P(O_2 = w, O_3 = d \mid S_1 = r)$

Hindsight problem continued...



- Just need to compute $P(O_2 = w, O_3 = d \mid S_1 = r)$
- $P(O_2 = w, O_3 = d \mid S_1 = r) =$
 $\sum_{s_2} P(S_2 = s_2, O_2 = w, O_3 = d \mid S_1 = r) =$
 $\sum_{s_2} P(S_2 = s_2 \mid S_1 = r) P(O_2 = w \mid S_2 = s_2) P(O_3 = d \mid S_2 = s_2)$
- First two factors directly in the model; last factor is a “smaller” problem of the same kind
- Use dynamic programming, backwards from the future
 - Similar to forwards approach from the past

Backwards reasoning in general

- Want to know $P(O_{k+1} = o_{k+1}, \dots, O_t = o_t \mid S_k)$

- First compute

$$P(O_t = o_t \mid S_{t-1}) = \sum_{s_t} P(S_t = s_t \mid S_{t-1}) P(O_t = o_t \mid S_t = s_t)$$

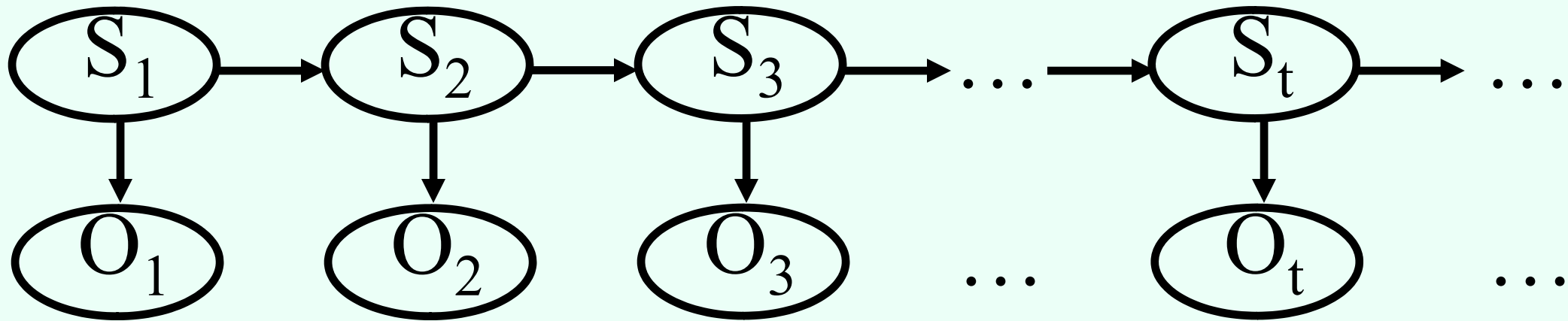
- Then compute

$$P(O_t = o_t, O_{t-1} = o_{t-1} \mid S_{t-2}) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1} \mid S_{t-2}) P(O_{t-1} = o_{t-1} \mid S_{t-1} = s_{t-1}) P(O_t = o_t \mid S_{t-1} = s_{t-1})$$

- Etc.

Variable elimination

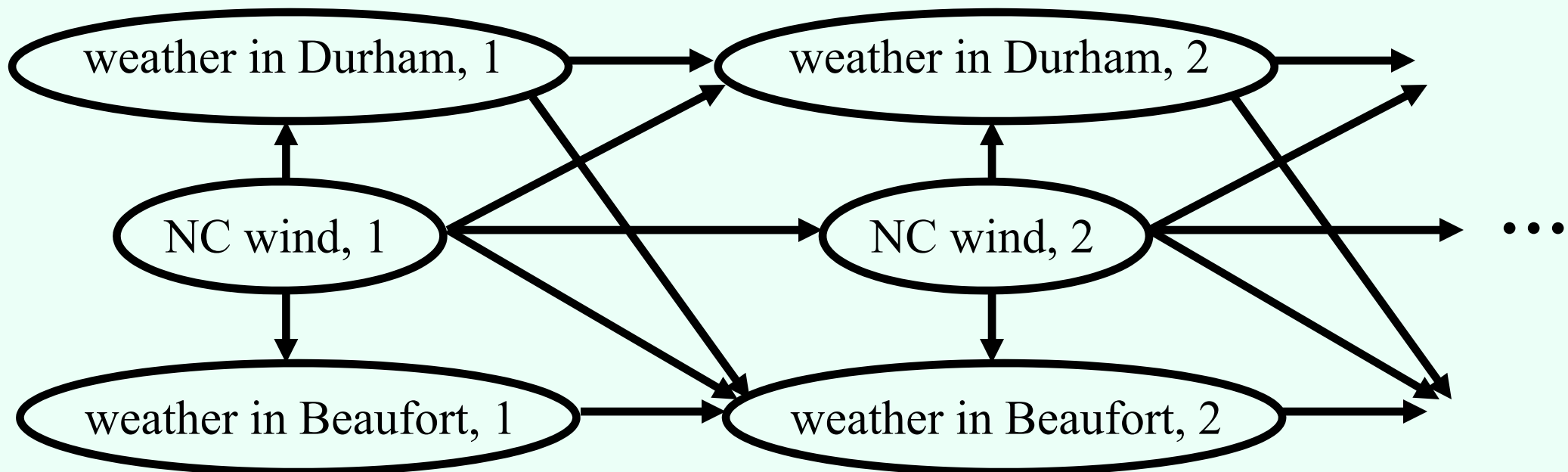
- Because all of this is inference in a Bayes net, we can also just do variable elimination



- E.g., $P(S_3 = r, O_1 = d, O_2 = w, O_3 = w) =$
 $\sum_{s_2} \sum_{s_1} P(S_1 = s_1) P(O_1 = d | S_1 = s_1) P(S_2 = s_2 | S_1 = s_1)$
 $P(O_2 = w | S_2 = s_2) P(S_3 = r | S_2 = s_2) P(O_3 = w | S_3 = r)$
- It's a tree, so variable elimination works well

Dynamic Bayes Nets

- So far assumed that each period has one variable for state, one variable for observation
- Often better to divide state and observation up into multiple variables



edges both within a period, and from one period to the next...

Some interesting things we skipped

- Finding the most likely **sequence** of states, given observations
 - Not necessary equal to the sequence of most likely states! (example?)
 - **Viterbi algorithm**
 - Key idea: for each period t , for every state, keep track of most likely sequence to that state at that period, given evidence up to that period
- Continuous variables
- Approximate inference methods
 - **Particle filtering**