

# **CPS 570: Artificial Intelligence**

## **Introduction to probability**

Instructor: Vincent Conitzer

# Uncertainty

- So far in course, everything deterministic
- If I walk with my umbrella, I **will not** get wet
- But: there is some chance my umbrella will break!
- Intelligent systems must take possibility of failure into account...
  - May want to have backup umbrella in city that is often windy and rainy
- ... but should not be excessively conservative
  - Two umbrellas not worthwhile for city that is usually not windy
- Need **quantitative** notion of uncertainty

# Probability

- Example: roll two dice
- **Random variables:**
  - $X$  = value of die 1
  - $Y$  = value of die 2
- Outcome is represented by an ordered pair of values  $(x, y)$ 
  - E.g.,  $(6, 1)$ :  $X=6, Y=1$
  - **Atomic event** or **sample point** tells us the **complete** state of the world, i.e., values of **all** random variables
- Exactly one atomic event will happen; each atomic event has a  $\geq 0$  probability; sum to 1
  - E.g.,  $P(X=1 \text{ and } Y=6) = 1/36$

Y

6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36
	1	2	3	4	5	6

X

- An **event** is a proposition about the state (=subset of states)
  - $X+Y = 7$
- Probability of event = sum of probabilities of atomic events where event is true

# Cards and combinatorics

- Draw a hand of 5 cards from a standard deck with  $4 \times 13 = 52$  cards (4 suits, 13 ranks each)
- Each of the  $\binom{52}{5}$  hands has same probability  $1/\binom{52}{5}$
- Probability of event = number of hands in that event /  $\binom{52}{5}$
- What is the probability that...
  - no two cards have the same rank?
  - you have a flush (all cards the same suit?)
  - you have a straight (5 cards in order of rank, e.g., 8, 9, 10, J, Q)?
  - you have a straight flush?
  - you have a full house (three cards have the same rank and the two other cards have the same rank)?

# Facts about probabilities of events

- If events  $A$  and  $B$  are disjoint, then
  - $P(A \text{ or } B) = P(A) + P(B)$
- More generally:
  - $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- If events  $A_1, \dots, A_n$  are disjoint and exhaustive (one of them must happen) then  $P(A_1) + \dots + P(A_n) = 1$ 
  - Special case: for any random variable,  $\sum_x P(X=x) = 1$
- Marginalization:  $P(X=x) = \sum_y P(X=x \text{ and } Y=y)$

# Conditional probability

- We might know something about the world – e.g., “ $X+Y=6$  or  $X+Y=7$ ” – given this (and **only** this), what is the probability of  $Y=5$ ?
- Part of the sample space is eliminated; probabilities are renormalized to sum to 1

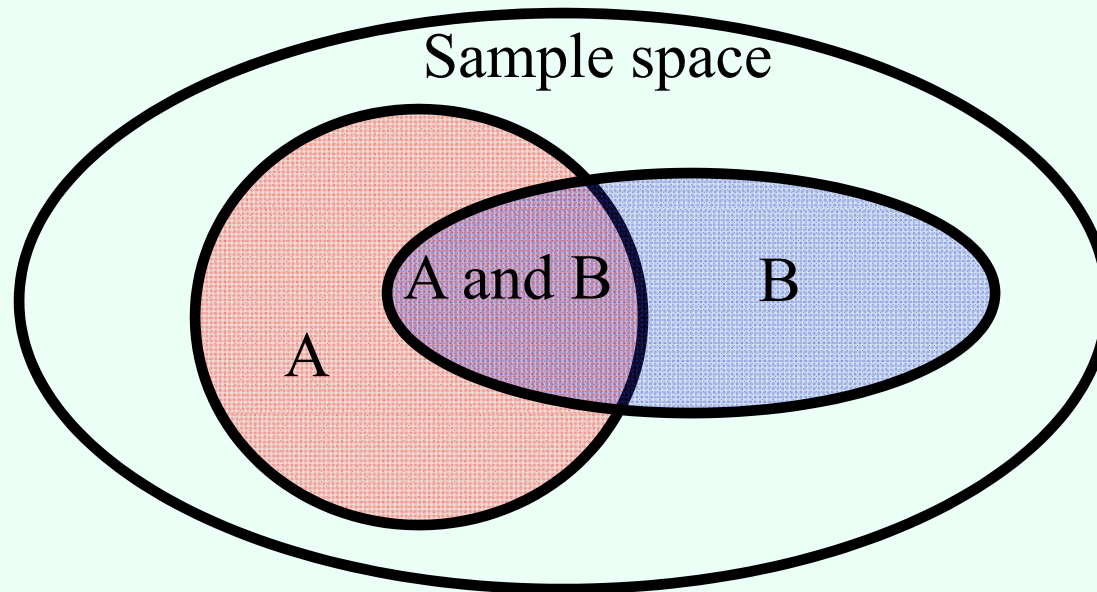
Y						
6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36
	1	2	3	4	5	6 X

Y						
6	1/11	0	0	0	0	0
5	1/11	1/11	0	0	0	0
4	0	1/11	1/11	0	0	0
3	0	0	1/11	1/11	0	0
2	0	0	0	1/11	1/11	0
1	0	0	0	0	1/11	1/11
	1	2	3	4	5	6 X

- $P(Y=5 \mid (X+Y=6) \text{ or } (X+Y=7)) = 2/11$

# Facts about conditional probability

- $P(A | B) = P(A \text{ and } B) / P(B)$



- $P(A | B)P(B) = P(A \text{ and } B) = P(B | A)P(A)$

- $P(A | B) = P(B | A)P(A)/P(B)$

- Bayes' rule

# Conditional probability and cards

- Given that your first two cards are Queens, what is the probability that you will get at least three Queens?
- Given that you have at least two Queens (not necessarily the first two), what is the probability that you have at least three Queens?
- Given that you have at least two Queens, what is the probability that you have three Kings?



# How can we scale this?

- In principle, we now have a complete approach for reasoning under uncertainty:
  - Specify probability for every atomic event,
  - Can compute probabilities of events simply by summing probabilities of atomic events,
  - Conditional probabilities are specified in terms of probabilities of events:  $P(A | B) = P(A \text{ and } B) / P(B)$
- If we have  $n$  variables that can each take  $k$  values, how many atomic events are there?

# Independence

- Some variables have nothing to do with each other
- Dice: if  $X=6$ , it tells us nothing about  $Y$
- $P(Y=y \mid X=x) = P(Y=y)$
- So:  $P(X=x \text{ and } Y=y) = P(Y=y \mid X=x)P(X=x) = P(Y=y)P(X=x)$ 
  - Usually just write  $P(X, Y) = P(X)P(Y)$
  - Only need to specify  $6+6=12$  values instead of  $6*6=36$  values
  - Independence among 3 variables:  $P(X, Y, Z) = P(X)P(Y)P(Z)$ , etc.
- Are the events “you get a flush” and “you get a straight” independent?

# An example without cards or dice

	Rain in Beaufort	Sun in Beaufort
Rain in Durham	.2	.1
Sun in Durham	.2	.5

*(disclaimer:  
no idea if  
these  
numbers are  
realistic)*

- What is the probability of
  - Rain in Beaufort? Rain in Durham?
  - Rain in Beaufort, given rain in Durham?
  - Rain in Durham, given rain in Beaufort?
- Rain in Beaufort and rain in Durham are **correlated**

# A possibly rigged casino

- With probability  $\frac{1}{2}$ , the casino is rigged and has dice that come up 6 only  $\frac{1}{12}$  of the time, and 1  $\frac{3}{12}$  of the time

$Z=0$  (fair casino)

Y	$Z=0$ (fair casino)					
6	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
5	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
4	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
3	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
2	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
1	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$
	1	2	3	4	5	6 X

$Z=1$  (rigged casino)

Y	$Z=1$ (rigged casino)					
6	$\frac{1}{96}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{144}$	$\frac{1}{288}$
5	$\frac{1}{48}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$
4	$\frac{1}{48}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$
3	$\frac{1}{48}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$
2	$\frac{1}{48}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{72}$	$\frac{1}{144}$
1	$\frac{1}{32}$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{96}$
	1	2	3	4	5	6 X

- What is  $P(Y=6)$ ?
- What is  $P(Y=6|X=1)$ ?
- Are they independent?

# Conditional independence

- Intuition:
  - the only reason that  $X$  tells us something about  $Y$ ,
  - is that  $X$  tells us something about  $Z$ ,
  - and  $Z$  tells us something about  $Y$
- If we already know  $Z$ , then  $X$  tells us nothing about  $Y$
- $P(Y | Z, X) = P(Y | Z)$  or
- $P(X, Y | Z) = P(X | Z)P(Y | Z)$
- “ $X$  and  $Y$  are conditionally independent given  $Z$ ”

# Medical diagnosis

- $X$ : does patient have flu?
- $Y$ : does patient have headache?
- $Z$ : does patient have fever?
- $P(Y,Z|X) = P(Y|X)P(Z|X)$
- $P(X=1) = .2$
- $P(Y=1 | X=1) = .5, P(Y=1 | X=0) = .2$
- $P(Z=1 | X=1) = .4, P(Z=1 | X=0) = .1$
- What is  $P(X=1|Y=1,Z=0)$ ?

*(disclaimer:  
no idea if  
these  
numbers are  
realistic)*

# Conditioning can also introduce dependence

- X: is it raining?
  - $P(X=1) = .3$
- Y: are the sprinklers on?
  - $P(Y=1) = .4$
  - X and Y are independent
- Z: is the grass wet?
  - $P(Z=1 \mid X=0, Y=0) = .1$
  - $P(Z=1 \mid X=0, Y=1) = .8$
  - $P(Z=1 \mid X=1, Y=0) = .7$
  - $P(Z=1 \mid X=1, Y=1) = .9$

		<i>Not wet</i>	
		Raining	Not raining
Sprinklers	No	.012	.056
	Yes	.054	.378

		<i>Wet</i>	
		Raining	Not raining
Sprinklers	No	.108	.224
	Yes	.126	.042

- Conditional on  $Z=1$ , X and Y are **not** independent
- If you know  $Z=1$ , rain seems likely; then if you also find out  $Y=1$ , this “explains away” the wetness and rain seems less likely

# Context-specific independence

- Recall  $P(X, Y | Z) = P(X | Z)P(Y | Z)$  really means: for all  $x, y, z$ ,
- $P(X=x, Y=y | Z=z) = P(X=x | Z=z)P(Y=y | Z=z)$
- But it may not be true for *all*  $z$
- $P(\text{Wet}, \text{RainingInLondon} | \text{CurrentLocation}=\text{New York}) = P(\text{Wet} | \text{CurrentLocation}=\text{New York})P(\text{RainingInLondon} | \text{CurrentLocation}=\text{New York})$
- **But not**
- $P(\text{Wet}, \text{RainingInLondon} | \text{CurrentLocation}=\text{London}) = P(\text{Wet} | \text{CurrentLocation}=\text{London})P(\text{RainingInLondon} | \text{CurrentLocation}=\text{London})$



# Pairwise independence does not imply complete independence

- $X$  is a coin flip,  $Y$  is a coin flip,  $Z = X \text{ xor } Y$
- Clearly  $P(X, Y, Z) \neq P(X)P(Y)P(Z)$
- But  $P(Z|X) = P(Z)$
- Tempting to say  $X$  and  $Y$  are “really” independent,  $X$  and  $Z$  are “not really” independent
- But:  $X$  is a coin flip,  $Z$  is a coin flip,  $Y = X \text{ xor } Z$  gives the exact same distribution

# Monty Hall problem

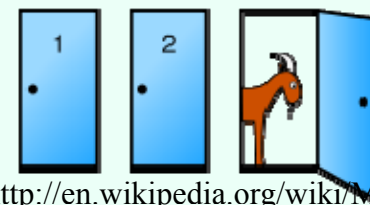


image taken from [http://en.wikipedia.org/wiki/Monty\\_Hall\\_problem](http://en.wikipedia.org/wiki/Monty_Hall_problem)

- Game show participants can choose one of three doors
- One door has a car, two have a goat
  - Assumption: car is preferred to goat
- Participant chooses door, but not opened yet
- At least one of the other doors contains a goat; the (knowing) host will open one such door (flips coin to decide if both have goats)
- Participant is asked whether she wants to switch doors (to the other closed door) – should she?

# Expected value

- If  $Z$  takes numerical values, then the **expected value** of  $Z$  is  $E(Z) = \sum_z P(Z=z) * z$ 
  - Weighted average (weighted by probability)
- Suppose  $Z$  is sum of two dice
- $E(Z) = (1/36)*2 + (2/36)*3 + (3/36)*4 + (4/36)*5 + (5/36)*6 + (6/36)*7 + (5/36)*8 + (4/36)*9 + (3/36)*10 + (2/36)*11 + (1/36)*12 = 7$
- Simpler way:  $E(X+Y)=E(X)+E(Y)$  (always!)
  - **Linearity of expectation**
- $E(X) = E(Y) = 3.5$

# Linearity of expectation...

- If  $a$  is used to represent an atomic state, then  $E(X) = \sum_x P(X=x) \cdot x = \sum_x (\sum_{a: X(a)=x} P(a)) \cdot x = \sum_a P(a) \cdot X(a)$
- $E(X+Y) = \sum_a P(a) \cdot (X(a)+Y(a)) = \sum_a P(a) \cdot X(a) + \sum_a P(a) \cdot Y(a) = E(X)+E(Y)$

# What is probability, anyway?

- Different philosophical positions:
- **Frequentism**: numbers only come from repeated experiments
  - As we flip a coin lots of times, we see experimentally that it comes out heads  $\frac{1}{2}$  the time
  - Problem: for most events in the world, there is no history of **exactly** that event happening
    - Probability that the Democratic candidate wins the next election?
- **Objectivism**: probabilities are a real part of the universe
  - Maybe true at level of quantum mechanics
  - Most of us agree that the result of a coin flip is (usually) determined by initial conditions + classical mechanics
- **Subjectivism**: probabilities merely reflect agents' beliefs