Due on September 23, 2015

Problem 1 [15 points]
In lecture, we discussed an algorithm for minimum-cost circulation by canceling minimum-mean cost cycles. Give an $O(mn)$ time algorithm to find a minimum-mean cost cycle.

(**Hint:** *You can modify the Bellman-Ford shortest paths algorithm.*)

Problem 2 [5 + 20 points]
The *transportation problem* is a generalization of the minimum-cost bipartite matching problem. You are given a bipartite graph $G = (A \cup B, E)$, edge costs $c : E \to \mathbb{N}$, supplies $s : A \to \mathbb{N}_{\geq 0}$, and demands $d : B \to \mathbb{N}_{\geq 0}$, such that the total supply equals the total demand.

$$\sum_{a \in A} s(a) = \sum_{b \in B} d(b) = U$$

A feasible solution is a transportation $f : E \to \mathbb{N}_{\geq 0}$ of supplies to demands, i.e.:

$$\begin{aligned} \sum_{b \in B} f(a,b) &= s(a) \qquad \forall a \in A \\ \sum_{a \in A} f(a,b) &= d(b) \qquad \forall b \in B \end{aligned}$$

The cost of a transportation f is:

$$c(f) = \sum_{e \in E} c(e) f(e)$$

The transportation problem asks to find a transportation of minimum cost.

(a) Show that transportation is a special case of minimum-cost flow.

(b) Give an $O(|V| + |E|)$ time reduction from minimum-cost flow to the transportation problem. That is, describe the transformation from a min-cost flow instance to a transportation instance, and show how a transportation solution can be bijectively mapped to a min-cost flow solution of the same value.

(**Hint:** *The transportation problem has no edge capacities. How can the MCF capacity constraints be expressed as transportation supply/demand constraints?*)

Problem 3 [15 points]
An *s-t vertex cut* is defined as a subset of vertices whose removal disconnects vertices s and t. Given a unit-capacity, undirected graph G and two vertices s and t, we want to show that the maximum number of vertex-disjoint s-t paths is equal to the minimum size of an s-t vertex cut. To show this, formulate the problems as a primal-dual LP pair and show that both LPs have integral optimal solutions.

(**Hint:** *To show integrality of the disjoint paths LP, you can use the fact that the maxflow LP is integral in a graph with integer edge capacities.*)

Problem 4 [5 + 5 + 5 + 5 points]
In this problem, you need to prove the integrality of the maximum bipartite matching LP (shown below) using a different approach from that we saw in class. Denote the bipartite graph $G = (A \cup B, E)$, where A and B are the two sets of vertices and E is a set of edges between A, B. Note that the bipartite graph is unweighted.

$$\max \sum_{e \in E} x_e$$
$$\text{s.t.} \sum_{e: v \in e} x_e \leq 1 \quad \forall v \in A \cup B$$
$$x_e \geq 0 \quad \forall e \in E$$

(a) Show that the number of edges with $x_e > 0$ in a basic feasible solution of the LP is at most $|A| + |B|$. Deduce that one of the following holds: (1) there exists an edge e with $x_e = 0$; (2) there exists a vertex of degree less than 2; (3) all vertices have degree exactly 2.

(b) If there is some vertex whose degree is less than 2, show that it can be removed from consideration in the LP. Describe the removal step and argue its correctness.

(c) If all vertices have degree exactly 2, show that the graph is a collection of disjoint even-length cycles. Find an integral matching of size at least the optimal LP value in such a graph.

(d) Use the above arguments to show that the bipartite matching LP is integral. Argue that an integral solution can be obtained by solving at most $\min(|A|, |B|)$ LPs (rather than $|E|$ LPs as shown in class).

Problem 5 [10 + 20 points]
This problem is about the minimum spanning tree (MST) problem in an undirected graph, where every edge has a non-negative cost.

(a) Write an LP for the MST problem, which enforces the constraint that at least one edge is selected from every cut. Show that you can solve this LP in polynomial time by using an efficient separation oracle. (It is sufficient to identify the problem you need to solve in the separation oracle; you do not need to give an explicit algorithm.) Give an example to show that this LP is not integral.

(b) Write a different LP for the MST problem where you enforce an upper bound on the number of edges selected from any induced subgraph, and a lower bound on the total number of edges. Show that any integral feasible solution to this LP is a spanning tree of the graph.