

**Due on November 18, 2015**

**Problem 1 [10 + 10 + 10 + 20 points]**

Consider the problem of online scheduling on identical, parallel machines.

- We have  $m$  machines  $i$ , and  $n$  jobs  $j$  arriving online.
- Each job  $j$  has a processing time  $p_j \in \mathbb{N}$ .
- Jobs arrive one by one, and must be assigned before the next job arrives.

A machine's completion time is the sum of processing times of all jobs assigned to it. The objective is to minimize the *makespan*, the maximum completion time of any machine.

- Consider the following greedy algorithm: when job  $j$  arrives, assign it to the (currently) least loaded machine. Prove this greedy algorithm has competitive ratio 2.
- Show that this analysis is asymptotically tight by giving an example where the competitive ratio approaches 2 as the number of jobs and/or machines increases.

Now, consider a variant of the problem where each job can only be processed to a subset of machines  $S_j \subseteq \{1, \dots, m\}$ , and all  $p_j = 1$ .

- Show that no deterministic algorithm has competitive ratio better than  $\Omega(\log m)$ .  
(**Hint:** Use the adversary model, and for each job pick subsets with  $|S_j| = 2$ .)
- The earlier greedy algorithm generalizes to this version: for each job  $j$ , assign  $j$  to the least loaded machine in  $S_j$ . Show that this greedy algorithm has competitive ratio  $O(\log m)$ .  
(**Hint:** Let  $\text{OPT}$  be the optimal offline makespan. There are at most  $n \leq m\text{OPT}$  jobs. How many jobs can be scheduled to time  $> \text{OPT}$ ? How many jobs can be scheduled to time  $> 2\text{OPT}$ ?)

**Problem 2 [5 + 5 + 20 points]**

One of the LP relaxations for MST we discussed in class uses constraints on cuts with undirected edges.

$$\begin{aligned} & \min \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e: (S, \bar{S})} x_e \geq 1 \quad \forall S \subset V \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

- (a) Prove that the integrality gap of the undirected cut relaxation of MST is 2, as we did during lecture.

A different LP relaxation also places constraints on cuts, but uses variables on directed edges. The input is still an undirected graph, but we have two directed variables  $y_{uv}, y_{vu}$  for each edge  $(u, v)$ . We pick an arbitrary vertex  $r \in V$  as the *root*

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x_e = y_{uv} + y_{vu} \quad \forall e = (u, v) \in E \\ & \sum_{\substack{u \in S \\ v \notin S}} y_{uv} \geq 1 \quad \forall S : S \subset V, r \notin S \\ & x_e \geq 0 \quad \forall e \in E \\ & y_{uv} \geq 0 \quad \forall u, v \in V \end{aligned}$$

The directed variables of this LP maintain a directed spanning tree with edges pointed towards the root.

- (b) Show that this is a valid linear program for minimum spanning tree. In other words, show that every MST satisfies the constraints of this program.
- (c) Show that the integrality gap for the directed cut relaxation of MST is 1, by constructing and analyzing a primal-dual algorithm of this LP.

**(Hint:** *You need to use recursion in the primal dual algorithm. Think of every undirected edge as two directed edges. Consider a vertex  $v$  other than  $r$ . Add the minimum cost edge leaving  $v$  to the MST. Reduce the cost of every other edge leaving  $v$  by the cost of the edge you just added to the MST, and contract the edge that you just added. Recurse on this new graph. How do you construct your dual solution in the original graph using the dual from the recursive subproblem? Along the way, you might need to bound the outdegree of every vertex in the directed set of edges you are added to the MST.*)

**Problem 3 [0 points]**

What did you like most about the course? What did you like the least?