COMPSCI 532: Design and Analysis of Algorithms	September 11, 2015

Lecture 6

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1 Overview

In this lecture, we introduce linear programming, LP duality, Farkas' Lemma, and complementary slackness. Linear programs are simply constrained optimization problems where all functions are linear. Even when limiting ourselves to linear functions, such problems are amazingly expressive. Many of the combinatorial problems we have seen until now can be expressed as linear programs, so there will be many examples.

Duality and complementary slackness are also the foundation behind the *primal-dual* algorithmic framework. We will return to this idea later in the course.

2 Linear Programs

A linear program is a twist on the constraint satisfaction problem, which seeks an assignment of variables optimizing an *objective* subject to *constraints*.

$$\begin{aligned} \min/\max \quad & f(x) \\ \text{s.t.} \quad & g_1(x) \le b_1 \\ & g_2(x) = b_2 \\ & g_3(x) \ge b_23 \\ & \dots \end{aligned}$$

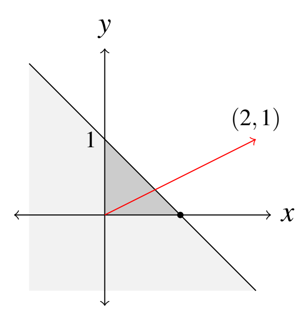
In linear programming, both objective and constraints are *linear functions* of variables.

$$f(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

With n variables, we can visualize (the solutions of) any linear program as a *convex polyhedron* in $\mathbb{R}^n$.

Example 1. Consider the following linear program:

$$\begin{aligned} \max \quad & 2x + y \\ \text{s.t.} \quad & x + y \le 1 \\ & x,\ y \ge 0 \end{aligned}$$



Each complete assignment of variables is a point in $\mathbb{R}^2$.

$$x = 0,\ y = 1 \to (0,1)$$

The linearity of the objective function makes its coefficients appear as a direction in $\mathbb{R}^2$. The linear program asks to find a feasible point furthest in the direction of the objective. This can be evaluated as the dot product of both vectors, which produces the original objective expression.

$$(2,1)^{\mathsf{T}}(x,y) = 2x+y$$

The linearity of constraints causes them to appear as *half-spaces* in $\mathbb{R}^2$, bounded by *hyperplanes*. Points within a half-space "satisfy" the constraint, while points on the bounding hyperplane meet that constraint with equality. Points in the intersection of every half-spaces are *feasible* solutions. The intersection of the half-spaces forms a convex polyhedron. Thus, linear programming is a special case of convex programming.

Another way to interpret the objective is as the direction of "gravity". If we drop a ball from the inside of the feasible polyhedron, the point it stops at is the optimal. Of course, this suggests several possibilities for the solution. We might have a finite (bounded) solution on some intersection of constraints, an infinite (unbounded) solution if the polyhedron has no "bottom", and finally no solution if the feasible space is empty and we cannot initially place the ball. We will state these possibilities formally later.

Example 2. Maximum flow can be expressed as a linear program. Recall that $f(v,w)$ is the *net flow* across (v,w). The variables in this program are the raw flows $r(v,w)$.

$$\begin{array}{lll} \max & \sum_w r(s,w) - \sum_w r(w,s) & \\ \text{s.t.} & r(v,w) \le u(v,w) & \forall (v,w) \\ & \sum_w r(v,w) - \sum_w r(w,v) = 0 & \forall v \ne s,t \\ & r(v,w) \ge 0 & \forall (v,w) \end{array}$$

Example 3. A linear program for s-t shortest paths. We have a variable $d(v)$ for each vertex v.

$$\begin{array}{lll} \max & d(s) & \\ \text{s.t.} & d(t) = 0 & \\ & d(v) \le d(w) + 1 & \forall (v,w) \in E \\ & d(v) \ge 0 & \end{array}$$

$d(v)$ act as a lower bound on the true distance from t.

Linearity allows us to represent linear programs in a compact matrix form. We make a vector of variables x and objective coefficients c. Each constraint is a row of matrix A bounded by a value in b.

Definition 1. *Let $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The* ***canonical forms*** *of a linear program are:*

$$\begin{array}{ll} \max & c^{\mathsf{T}}x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array} \qquad\qquad \begin{array}{ll} \min & c^{\mathsf{T}}x \\ \text{s.t.} & Ax \ge b \\ & x \ge 0 \end{array}$$

Claim 1. *Any linear program may be represented in canonical form.*

Proof. Where might a linear program deviate from canonical form? Consider a minimization problem with constraints A, where a_j is the jth row of A.

1. A variable x may be constrained to be negative ($x \leq 0$). We can perform a change of variables using $z = -x$. It follows that $z \geq 0$, which fits the canonical form sign constraint.

2. A variable x may not be constrained in sign at all. We can again perform a change of variables, this time using $z_1 - z_2 = x$, where $z_1, z_2 \geq 0$. Intuitively, we can represent any number as the difference of two nonnegative numbers.

3. A constraint may be in the $\leq$ direction rather than the $\geq$ direction.

$$a_j x \leq b_j$$

Negating the entire constraint gives us the correct direction.

$$-a_j x \geq -b_j$$

So we will replace the row a_j and b_j with their negatives.

4. A constraint may be with equality.

$$a_j x = b_j$$

Recall that equality holds when both $\leq$ and $\geq$ hold. We replace the equality constraint with two inequalities:

$$\begin{aligned} & a_j x = b_j \\ \Longrightarrow \quad & (a_j x \geq b_j) \text{ and } (a_j x \leq b_j) \\ \Longrightarrow \quad & (a_j x \geq b_j) \text{ and } (-a_j x \geq -b_j) \end{aligned}$$

5. Although not applicable to canonical form, if we want to transform a $\geq$ constraint into an equality constraint, we can accomplish this by adding a slack variable $z \geq 0$:

$$\begin{aligned} & a_j x \geq b_j \\ \Longrightarrow \quad & a_j x - z = b_j \end{aligned}$$

□

Claim 2. *The feasible space for a linear program is convex. That is, if x_1, x_2 are two feasible solutions, then any convex combination ($\theta \in [0,1]$) is also a solution:*

$$\theta x_1 + (1-\theta) x_2$$

Of course, this works for convex combinations of any number of solutions.

Proof. Intuitively, the half-spaces defined by linear constraints are convex sets, so their intersection is convex. For a minimization problem in standard form, let x_1, x_2 as above and $\bar{x} = \theta x_1 + (1-\theta) x_2$. Obviously, $\bar{x} \geq 0$. Additionally:

$$\begin{aligned} A\bar{x} &= A(\theta x_1 + (1-\theta) x_2) \\ &= \theta A x_1 + (1-\theta) A x_2 \\ &\leq \theta b + (1-\theta) b \\ &= b \end{aligned}$$

Thus, $\bar{x}$ is feasible. □

2.1 Primal-Dual Pairs

Suppose we have the following minimization program:

$$\begin{aligned} \min \quad & c^{\intercal}x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Let the optimal solution be x^*. It's straightforward to show that the value of the optimal solution is below some threshold α: demonstrate any feasible x with value α. The optimal must do at least as well.

$$c^{\intercal}x = \alpha \implies c^{\intercal}x^* \leq \alpha$$

The opposite direction is not as easy – how would one show that $c^{\intercal}x^* \geq \alpha$? We cannot use the same trick of finding a solution of value α. Suppose instead we find $y \geq 0$ such that:

$$A^{\intercal}y \leq c$$

Claim 3. *$b^{\intercal}y$ is a lower bound on $c^{\intercal}x^*$.*

Proof. By feasibility:

$$Ax^* \geq b$$

Left multiply by $y^{\intercal}$.

$$y^{\intercal}Ax^* \geq y^{\intercal}b$$

Apply our assumption on $y^{\intercal}A$.

$$c^{\intercal}x^* \geq y^{\intercal}b = b^{\intercal}y$$

□

Clearly, the larger the value of $b^{\intercal}y$, the better the bound is. The constraints on y form a new linear program:

$$\begin{aligned} \max \quad & b^{\intercal}y \\ \text{s.t.} \quad & A^{\intercal}y \leq c \\ & y \geq 0 \end{aligned}$$

We formalize this notion as LP *duality*.

Definition 2. *For a **primal** (P) linear program in the form:*

$$\begin{aligned} \min \quad & c^{\intercal}x \\ \textit{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

*The **dual** (D) linear program is:*

$$\begin{aligned} \max \quad & b^{\intercal}y \\ \textit{s.t.} \quad & A^{\intercal}y \leq c \\ & y \geq 0 \end{aligned}$$

If the primal is a canonical form maximization problem, swap P and D above.

Primal (minimize)	*Dual (maximize)*
jth constraint $\geq$	jth variable ≥ 0
jth constraint $\leq$	jth variable ≤ 0
jth constraint $=$	jth variable unrestricted
ith variable ≥ 0	ith constraint $\leq$
ith variable ≤ 0	ith constraint $\geq$
ith variable unrestricted	ith constraint $=$

Figure 3: LP conversion table from Chapter 4 of [BHM77].

Fact 4. *If D is the dual of P, then the dual of D is P. "Dual of the dual is the primal."*

Each primal variable begets a dual constraint, and each primal constraint begets a dual variable. If the program is not in canonical form, we can convert to canonical form, take the dual, and then convert back. See the Figure 3 for general duality rules derived from performing this exercise.

Example 4. We repeat the argument from before for constructing the dual, but with an explicit example. Suppose we have the following primal linear program, and again want a lower bound on $c^\mathsf{T}x^*$.

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_1 + 3x_2 \geq 3 \\ & x_1 \geq 1 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

We can combine the first constraint and the third constraint to produce the objective (and a bound on it).

$$\begin{aligned} (-1)(x_1 + x_2) + 3(x_1) &\geq (-1)(2) + 3(1) \\ 2x_1 + x_2 &\geq 4 \end{aligned}$$

We can do the same thing with the second and third constraints to obtain another bound.

$$\begin{aligned} -\frac{1}{3}(x_1 + 3x_2) + \frac{7}{3}(x_1) &\geq -\frac{1}{3}(3) + \frac{7}{3}(1) \\ 2x_1 + x_2 &\geq \frac{4}{3} \end{aligned}$$

So what is the best bound? We want to find the best linear combinations of the primal constraints. To avoid flipping the inequalities, we will only allow coefficients where $y \geq 0$.

$$y_1(x_1 + x_2) + y_2(x_1 + 3x_2) + y_3(x_1) \geq y_1(2) + y_2(3) + y_3(1)$$

Then, our dual objective to maximize is the right hand side.

$$\max\{2y_1 + 3y_2 + y_3\}$$

But we need the left hand side coefficients to be no more than the coefficients of the primal objective.

$$\begin{aligned} y_1(x_1) + y_2(x_1) + y_3(x_1) &\leq 2x_1 \\ y_1(x_2) + y_2(3x_2) + y_3(0) &\leq -x_2 \end{aligned}$$

Factoring out the x variables, we have constraints on y:

$$\begin{aligned} y_1 + y_2 + y_3 &\leq 2 \\ y_1 + 3y_2 &\leq -1 \end{aligned}$$

Putting them together, we have our dual program.

$$\begin{aligned} \max \quad & 2y_1 + 3y_2 + y_3 \\ \text{s.t.} \quad & y_1 + y_2 + y_3 \leq 2 \\ & y_1 + 3y_2 \leq -1 \\ & y_1,\ y_2,\ y_3 \geq 0 \end{aligned}$$

Hopefully, this exercise helped show how dual variables are generated by primal constraints, and dual constraints are generated by primal variables. One can also verify that the matrix notation in Definition 2 matches with the dual constructed here.

3 Weak and Strong Duality

The claim we proved earlier, that the maximization dual *lower bounds* the minimization primal, is also known as *weak duality*.

Theorem 5. *Let x be any feasible primal solution, y be any feasible dual solution. The* ***weak duality*** *theorem states that, for minimization primal, maximization dual:*

$$c^\intercal x \geq b^\intercal y$$

Alternatively, for maximization primal, minimization dual:

$$c^\intercal x \leq b^\intercal y$$

Proof. Same as the proof for Claim 3. You may have noticed by now that most of our results have mirrored versions for when primal is minimization or maximization. These are more or less exchangeable (we can always switch to the dual problem as our "primal", as suggested by Fact 4). $\square$

Example 5. A different linear program for flow based on flow decomposition.

Let P be the s-t paths in the flow decomposition.

$$\begin{aligned} \max \quad & \sum_{p \in P(s,t)} f(p) \\ \text{s.t.} \quad & \sum_{p:(v,w) \in p} f(p) \leq u(v,w) \quad \forall (v,w) \in E \\ & f(p) \geq 0 \end{aligned}$$

The dual is:

$$\begin{aligned} \min \quad & \sum_{(v,w) \in E} u(v,w)\ell(v,w) \\ \text{s.t.} \quad & \sum_{(v,w) \in p} \ell(v,w) \geq 1 \qquad \forall p \in P(s,t) \\ & \ell(v,w) \geq 0 \end{aligned}$$

One can interpret this as: the length function with minimum volume such that $d_\ell(s,t) \geq 1$. For any s-t cut $(S,\overline{S})$,

$$\ell(v,w) = \begin{cases} 1 & \text{if } (v,w) \in (S,\overline{S}) \\ 0 & \text{otherwise} \end{cases}$$

is feasible, and therefore the max-flow $\leq$ min-cut by merely applying weak duality.

Corollary 6. *For primal-dual pair P and D, either of the following must hold.*

1. *P and D are feasible and bounded.*
2. *One of P and D is feasible but unbounded; then the other must be infeasible.*
3. *Both P and D are infeasible.*

When the first case holds, we can say something stronger. *Strong duality* applies.

Theorem 7. *Let P and D be a primal-dual pair. The* ***strong duality*** *theorem states that the following are equivalent:*

1. x^*, y^* *are finite optimal solutions to P and D respectively.*
2. $c^\intercal x^* = b^\intercal y^*$

To prove this theorem, we first need a few more tools. The next few theorems will use LP primal and dual in *standard form*.

Definition 3. *The* ***standard form*** *of a minimization problem and its dual is:*

$$\begin{array}{llcll} & \text{(P)} & \qquad & & \text{(D)} \\ \min & c^\intercal x & & \max & b^\intercal y \\ \text{s.t.} & Ax = b & & \text{s.t.} & A^\intercal y \leq c \\ & x \geq 0 & & & \end{array}$$

As with canonical form, all linear programs can be represented in this form, so theorems proved using standard form will apply to any linear program.

Note that the dual variables y are unrestricted in sign.

3.1 Farkas' Lemma

We state the following theorem without proof, and use it to prove Farkas' Lemma. The proof is straightforward, but requires definition of convex cones.

Theorem 8. *If $p \in \mathbb{R}^n$ is not in closed convex set $C \subseteq \mathbb{R}^n$, there exists a* ***separating hyperplane*** *which separates p from C. In other words, we can find a vector $v \in \mathbb{R}^n$ and real α where:*

$$\begin{array}{rcll} q^\intercal v & \leq & \alpha & \qquad \forall q \in C \\ p^\intercal v & > & \alpha & \end{array}$$

Rearranging:

$$p^\intercal v > q^\intercal v \qquad \forall q \in C$$

Theorem 9. ***Farkas' Lemma**, or the Farkas Alternative, states that exactly one of the following sets is nonempty:*

1. $\{x \mid Ax = b,\ x \geq 0\}$
2. $\{y \mid A^\intercal y \leq 0,\ y^\intercal b > 0\}$

x and y are vectors, so when we say $x \geq 0$ (for example), we mean that every scalar entry is nonnegative.

Proof. We prove that (1) $\implies \neg$(2), and that $\neg$(1) $\implies$ (2)

1. Suppose (1) is feasible. If (2) is also feasible, then some y from the set achieves:

$$\begin{aligned} Ax &= b \\ y^\intercal Ax &= y^\intercal b > 0 \end{aligned}$$

However, we know that the objective for P has:

$$c^\intercal x^* \leq c^\intercal (0) = 0$$

Thus:

$$c^\intercal x^* < y^\intercal b$$

This violates weak duality of the pair, a contradiction. (2) must not be feasible.

2. If (1) is not feasible, then by definition:

$$b \notin \{Ax \mid x \geq 0\}$$

Applying the separating hyperplane theorem, there exists y where:

$$y^\intercal b > \sup_{x \geq 0} \{y^\intercal (Ax)\}$$

Transposing both sides:

$$b^\intercal y > \sup_{x \geq 0} \{x^\intercal A^\intercal y\} \geq 0$$

The second half is at least 0, e.g. using $x = 0$. If $A^\intercal y > 0$, then $x^\intercal A^\intercal y$ is unbounded (x is nonnegative). Since $b^\intercal y$ is finite, this cannot be the case. It must be that:

$$A^\intercal y \leq 0$$

Then (2) is feasible (both conditions were met above).

□

3.2 Proof of Strong Duality

Finally, we prove strong duality using Farkas' Lemma.

Proof. We prove both directions.

1. (1) $\implies$ (2)

   We will use a matrix augmentation trick to make the Farkas conditions look more like the dual feasibility conditions. For $\varepsilon > 0$ and $\lambda \in \mathbb{R}$:

$$\hat{y} = \begin{bmatrix} y \\ \lambda \end{bmatrix} \quad \hat{A} = \begin{bmatrix} A \\ -c^\intercal \end{bmatrix} \quad \hat{b} = \begin{bmatrix} b \\ -(c^\intercal x^* - \varepsilon) \end{bmatrix}$$

   And x is unchanged. The matrix products now give us:

$$\begin{aligned} \hat{A}^\intercal \hat{y} &= A^\intercal y - c \\ \hat{b}^\intercal \hat{y} &= b^\intercal y - (c^\intercal x^* - \varepsilon) \\ \hat{A}x &= \begin{bmatrix} Ax \\ -c^\intercal x \end{bmatrix} \end{aligned}$$

   When we use $\varepsilon > 0$, the first Farkas conditions become:

$$\hat{A}x = \hat{b} \implies \begin{cases} Ax = b \quad \text{and} \\ c^\intercal x = c^\intercal x^* - \varepsilon \end{cases}$$

   This holds for no x, since x^* is optimal for minimization. Farkas tells us the second alternative holds, namely that there are $\hat{y}$ where:

$$\begin{aligned} \hat{A}^\intercal \hat{y} \le 0 &\implies A^\intercal y \le \lambda c \\ \hat{b}^\intercal \hat{y} > 0 &\implies b^\intercal y > \lambda (c^\intercal x^* - \varepsilon) \end{aligned}$$

   If $\lambda > 0$, dividing by λ gives us a feasible dual solution which is arbitrarily close to the primal optimal. Taking the limit would immediately show that $b^\intercal y^* = c^\intercal x^*$.

$$\begin{aligned} A^\intercal (y/\lambda) &\le c \\ b^\intercal (y/\lambda) &> c^\intercal x^* - \varepsilon \end{aligned}$$

   We will show that $\lambda > 0$ is indeed the case.

   Now replace $\varepsilon = 0$ in the augmented matrices. We recover:

$$\hat{A}x = \hat{b} \implies \begin{cases} Ax = b \quad \text{and} \\ c^\intercal x = c^\intercal x^* \end{cases}$$

   x^* satsifies these conditions by feasibility/optimality, so the first Farkas alternative holds. Therefore no y satisfy:

$$\begin{aligned} A^\intercal y &\le \lambda c \\ b^\intercal y &> \lambda c^\intercal x^* \end{aligned}$$

This includes the $\hat{y}$ (for $\varepsilon > 0$) we found previously. The only difference, though, is in the second condition. For one of those $\hat{y}$ and $\varepsilon > 0$, it must be that:

$$\begin{aligned} b^\intercal y &> \lambda(c^\intercal x^* - \varepsilon) \\ b^\intercal y &\leq \lambda c^\intercal x^* \end{aligned}$$

Rearranging:

$$\begin{aligned} & \lambda c^\intercal x^* > \lambda(c^\intercal x^* - \varepsilon) \\ \implies & 0 > -\lambda\varepsilon \\ \implies & \lambda > 0 \end{aligned}$$

We can now divide by λ without flipping the inequalities. There exists a $y' = y/\lambda$ where, for $\varepsilon > 0$:

$$\begin{aligned} A^\intercal y' &\leq c \\ c^\intercal x^* \geq b^\intercal y' &> c^\intercal x^* - \varepsilon \end{aligned}$$

The limit as $\varepsilon \to 0$ is $b^\intercal y' = c^\intercal x^*$. The feasible space for D is compact and $b^\intercal y^*$ is its supremum, so y^* achieves this limit.

$$c^\intercal x^* = b^\intercal y^*$$

2. (2) $\implies$ (1)

   This direction follows immediately from weak duality. For all feasible x, y:

$$c^\intercal x \geq b^\intercal y$$

   x^*, y^* satisfy this with equality, so they must be optimal.

□

Example 6. We will show that the max-flow min-cut theorem can be proved with an application of strong duality. It is not enough to simply apply strong duality to the primal-dual pair, since we need the dual variables to be integral to be a real cut. If we can prove that an integral dual solution exists, strong duality directly gives us the theorem.

To prove that an integral solution exists, we will use *rounding*. Recall the dual program, which finds fractional min-cuts:

$$\begin{aligned} \min \quad & \sum_{(v,w)\in E} u(v,w)\ell(v,w) \\ \text{s.t.} \quad & \sum_{(v,w)\in p} \ell(v,w) \geq 1 \qquad \forall p \in P(s,t) \\ & \ell(v,w) \geq 0 \end{aligned}$$

Claim 10. *For any feasible $\ell(\cdot)$, there is $\ell'(\cdot) \in \{0,1\}^m$ such that $\sum u(e)\ell'(e) \leq \sum u(e)\ell(e)$. For an optimal $\ell^*(\cdot)$, there is an integral solution of equal value.*

Proof. We will provide a randomized algorithm for rounding any fractional solution $\ell(\cdot)$ to an integral $\ell'(\cdot)$ where:

$$\mathbb{E}\left[\sum_{e\in E} u(e)\ell'(e)\right] \leq \sum_{e\in E} u(e)\ell(e)$$

At least one randomized outcome is at least as low as the expectation, proving the claim.

We will use each $\ell'(v,w)$ as a 0-1 random variable. Let $d_\ell(v)$ be the shortest distance of v from t under edge lengths $\ell(\cdot)$. We will assign to each vertex a potential $d(v) := d_\ell(v)$. By the properties of shortest distances:

$$d(v) \leq d(w) + \ell(v,w) \qquad \forall (v,w) \in E$$

By the constraints, we must also have $d(s) = d_\ell(s) \geq 1$. To round, we choose $r \in (0,1)$ and let $S = \{v \mid d(v) \geq r\}$. If (v,w) crosses the S cut, we assign $\ell'(v,w) = 1$.

$$\Pr\left[\ell'(v,w) = 1\right] = \frac{d(v) - d(w)}{d(s)} \leq \frac{\ell(v,w)}{1}$$

Now in expectation:

$$\begin{aligned}
\mathbb{E}\left[\sum_{e\in E} u(e)\ell'(e)\right] &= \sum_{e\in E} u(e)\Pr\left[\ell'(e) = 1\right] \\
&\leq \sum_{e\in E} u(e)\ell(e)
\end{aligned}$$

As desired. □

4 Complementary Slackness

The final theorem we present is for *complementary slackness*. While the previous theorems controlled the values of solutions, complementary slackness deals with the values of individual variables.

Theorem 11. *For optimal solutions x^* (P) and y^* (D), either dual constraint i is tight, or $x_i^* = 0$. In other words,*

$$(c - A^\intercal y^*)^\intercal x^* = 0$$

Duality allows us make the same statement on primal constraints and indices of dual optimal y^*.

Proof. Using standard form, we know that feasible x satisfy:

$$Ax = b$$

Now,

$$\begin{aligned}
x^{*\intercal}(c - A^\intercal y^*) &= c^\intercal x^* - (y^{*\intercal} A)x^* \\
&= c^\intercal x^* - y^{*\intercal}(Ax^*) \\
&= c^\intercal x^* - y^{*\intercal} b \\
&= 0
\end{aligned}$$

The final equality holds by strong duality. □

Example 7. For this example, we will revisit minimum-cost circulation. Let $f(e)$ be a variable for the flow on e, $c(e)$ and $u(e)$ be the cost and capacity of e respectively.

$$\begin{aligned} \min \quad & \sum_{e \in E} c(e) f(e) \\ \text{s.t.} \quad & \sum_{x \in V} f(x,v) - \sum_{y \in V} f(v,y) = 0 && \forall v \in V \\ & f(v,w) \le u(v,w) && \forall (v,w) \in E \\ & f(v,w) \ge 0 && \forall (v,w) \in E \end{aligned}$$

Taking the dual, we have variables $\ell(e)$ for each edge (due to capacity constraints) and $p(v)$ each vertex (due to balance constraints).

$$\begin{aligned} \max \quad & \sum_{e \in E} u(e) \ell(e) \\ \text{s.t.} \quad & \ell(v,w) - p(v) + p(w) \le c(v,w) && \forall (v,w) \in E \\ & \ell(v,w) \le 0 && \forall (v,w) \in E \end{aligned}$$

We can rewrite the dual constraints to recover reduced cost:

$$\ell(v,w) \le c(v,w) + p(v) - p(w) = c_p(v,w)$$

Because we are maximizing (and coefficients $u(e) \ge 0$), we want $\ell(v,w)$ as close to its upper bound as possible. There are two possibilities: this constraint or the nonpositivity constraint.

$$\ell(v,w) = \min\{0, c_p(v,w)\}$$

Now, what does complementary slackness say about the dual? $\ell(v,w)$ selects one of the two values in that minimum, so:

1. $f(v,w) = 0$, or $\ell(v,w) = c_p(v,w)$ and therefore $c_p(v,w) \le 0$.
2. $f(v,w) = u(v,w)$, or $\ell(v,w) = 0$ and therefore $c_p(v,w) \ge 0$.

We can reinterpret these cases for the possible values of $f(v,w)$.

1. $f(v,w) = 0$, so only $(v,w) \in G_f$. By the previous case analysis, $c_p(v,w) \ge 0$ at optimality.
2. $0 < f(v,w) < u(v,w)$, so both $(v,w),(w,v) \in G_f$. Simultaneously $c_p(v,w) \ge 0, c_p(v,w) \le 0$, so it must be that $c_p(v,w) = 0$ at optimality.
3. $f(v,w) = u(v,w)$, so only $(w,v) \in G_f$. By the previous case analysis, $c_p(v,w) \le 0$, therefore $c_p(w,v) \ge 0$ at optimality.

Together, the complementary slackness conditions show that at optimality:

$$c_p(v,w) \ge 0 \qquad \forall (v,w) \in G_f$$

This is the same as the optimality condition we derived when discussing minimum-cost circulations.

5 Summary

In this lecture, we introduced linear programs, LP duality, and proved some of the primary theorems related to linear programs. We observed that linear programs could model some of our favorite problems. One thing we have not discussed: How do we solve linear programs?

Most of you have seen algorithms (i.e. simplex, ellipsoid) for linear programs by now. However, in worst case simplex is exponential time, and ellipsoid is weakly-polynomial. One of the major open questions of the time is whether or not LP has a strongly-polynomial algorithm.

References

[BHM77] S.P. Bradley, A.C. Hax, and T.L. Magnanti. *Applied Mathematical Programming*. Addison-Wesley Publishing Company, 1977.