

Lecture 18

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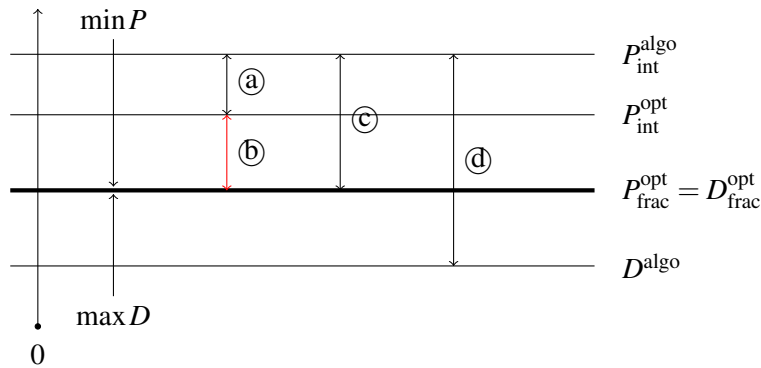
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# 1 Overview

In this note, we talk a bit more about integrality gaps, which we introduced in earlier lectures about approximation.

# 2 Integrality Gaps

Recall the approximation diagram for methods using LP relaxations (here, for a minimization problem):



The approximation factor is  $\textcircled{a}$ . It is often difficult to analyze this directly, so our previous techniques used the upper bound provided by  $\textcircled{c}$  (LP rounding) or  $\textcircled{d}$  (dual fitting and primal-dual). Both of these gaps, however, include the extra gap  $\textcircled{b}$ , which we call the **integrality gap**, which is the difference between the integer and fractional optimal solutions.

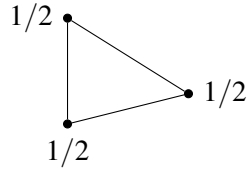
The integrality gap is a structural property of the LP, so we cannot avoid it if our approximation uses that particular LP relaxation. In fact, for most of the examples we have done, the approximation ratio we derived *was* the integrality gap (modulo constant factors).

## 2.1 Vertex cover

Recall the linear program relaxation for vertex cover:

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_v + x_w \geq 1 \quad \forall (v, w) \in E \\ & x_v \geq 0 \quad \forall v \in V \end{aligned}$$

In this case, the “relaxation” is the use of  $x_v \geq 0$ , rather than  $x_v \in \{0, 1\}$ . A very simple graph, the triangle, demonstrates that the optimal fractional solution beats the integer optimal.



Here, every integer solution must select at least 2 vertices to cover all three edges. The optimal fractional solution, however, is to place  $x_v = 1/2$  on all three vertices, with objective value  $3/2$ . The integrality gap is therefore:

$$\text{IG} \geq \frac{2}{3/2} = 4/3$$

We can extend this example to show that the integrality gap is arbitrarily close to 2. The triangle is a complete graph of 3 vertices,  $K_3$ . Consider instead the complete graph on  $n$  vertices,  $K_n$ .

- An integer solution must select at least  $n - 1$  vertices. If at least 2 vertices are unselected, then by completeness there is an edge between them, which is uncovered.
- Again, a feasible fractional solution is to assign  $x_v = 1/2$  to all vertices. The objective value is  $n/2$ .

For  $K_n$ , the integrality gap is:

$$\text{IG} \geq \frac{n-1}{n/2} = 2 \left(1 - \frac{1}{n}\right)$$

Then for any  $\varepsilon > 0$ , an example where the integrality gap is  $2 - \varepsilon$  is the complete graph on  $2/\varepsilon$  vertices. As  $n \rightarrow \infty$ , these examples have give a bound for the integrality gap which approaches 2. In the end, this makes a statement to the tune of: “It is impossible for a rounding or primal-dual algorithm to give an approximation ratio better than 2 for vertex cover.”

## 2.2 Minimum spanning tree

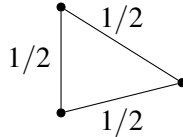
We previously discussed 2 programs for MST. This one (using partitions  $\pi$ ) had a primal-dual algorithm that we showed was equivalent to Kruskal’s algorithm, and is therefore exact.

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \pi} x_e \geq |\pi| - 1 \quad \forall \pi \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

Since the algorithm itself has an approximation ratio of 1, we know that the integrality gap must also be no more than 1 (no gap).

The other (weaker) linear program was as follows:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in (S, \bar{S})} x_e \geq 1 \quad \forall (S, \bar{S}) \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$



We will quantify this now by showing that this program has an integrality gap arbitrarily close to 2, as we did for vertex cover. Again, the triangle shows that the gap is  $> 1$ .

Any integer solution must select at least 2 edges to connect all 3 vertices. On the other hand, a valid fractional solution is to choose  $x_e = 1/2$  for every edge.

$$\text{IG} \geq \frac{2}{3/2} = \frac{4}{3}$$

This time, we find the extreme by examining the  $n$ -cycle. The triangle graph can be considered a cycle of 3 vertices. When there are  $n$ :

- An integral solution must select  $n - 1$  edges, otherwise those 2 edges form a cut of the cycle with no  $x_e > 0$  edges.
- A feasible fractional solution is to assign  $1/2$  to each edge, giving an objective value of  $n/2$ .

And we get the same integrality gap bound as in vertex cover:

$$\text{IG} \geq \frac{n-1}{n/2} = 2 \left( 1 - \frac{1}{n} \right)$$

Looking back at the partition-based LP, we can see that this solution is infeasible under the partition-based constraints.

### 3 Summary

It is not true in general that the approximation ratio of an algorithm is equal to the integrality gap of its relaxation. In fact, many programs have integrality gaps which are unknown. The proof outline which we have followed for the previous examples (provide an instance where the fractional solution is good and the integral one is not) is about as far as we know for deriving lower bounds on integrality gaps.

For proving hardness of approximation, however, more general techniques are known. Many of these rely on complexity assumptions, however, like  $P \neq NP$  or the even stronger *unique games conjecture*. These hardness results are not limited to the structure of the algorithm (e.g. whether a certain LP relaxation is used).