Due date: November 11, 2019

**Problem 1:** Let $G = (V, E)$ be an undirected graph where $|V| = n$, $|E| = m$. For $S \subseteq V$, the induced subgraph $G[S]$ is the graph $G[S] = (S, E')$ where $E' = \{(u, v) \in E \mid u, v \in S\}$.

Given a positive integer $k \leq n$, we define a $k$-dense induced subgraph of $G$ as $G[X] = (X, E')$ such that $|X| = k$ and $|E'| \geq mk(k - 1)/(n(n - 1))$. Give a randomized algorithm to find a $k$-dense subgraph. The running time of the algorithm should be polynomial with probability at least $1 - 1/n$.

**Problem 2:** Recall the maximum satisfiability problem, in which we are given $n$ boolean variables $x_1, \ldots, x_n$ and $m$ clauses, where each clause $C_j = \bigvee_{i \in P_j} x_i \lor \bigvee_{i \in N_j} \overline{x_i}$. Recall the linear program formulation of MAX-SAT:

$$\max \sum_{j=1}^m z_j$$

s.t. $\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \forall C_j$

$y_i \in \{0, 1\} \quad i = 1, \ldots, n$

$z_j \in \{0, 1\} \quad j = 1, \ldots, m$

Consider a randomized rounding algorithm that solves the relaxation of the above integer program and sets $y_i = 1$ ($x_i$ to true) with probability $y_i/2 + 1/4$. Show that this gives a randomized $3/4$-approximation algorithm for the maximum satisfiability problem.

**Problem 3:** In the maximum directed cut problem we are given a directed graph $G = (V, E)$. Each directed edge $(u, v) \in E$ has a nonnegative weight $w_{uv} \geq 0$. The goal is to partition $V$ into two sets $U$ and $W = V \setminus U$ such that we maximize the total weight of edges crossing from $U$ to $W$, i.e. maximize the sum of weights of edges $(u, v)$ where $u \in U$ and $v \in W$.

(i) Show that the following integer linear program models the maximum directed cut problem.

$$\max \sum_{(i,j) \in E} w_{ij} z_{ij}$$

s.t. $\quad z_{ij} \leq x_i \quad \forall (i, j) \in E$

$\quad z_{ij} \leq 1 - x_j \quad \forall (i, j) \in E$

$\quad x_i \in \{0, 1\} \quad \forall i \in V$

$\quad 0 \leq z_{ij} \leq 1 \quad \forall (i, j) \in E$

(ii) Consider a randomized rounding algorithm that solves the relaxation of the above integer program and puts vertex $i \in U$ with probability $x_i/2 + 1/4$. Show that this gives a randomized $1/2$-approximation algorithm for the maximum directed cut problem.
**Problem 4:** Let $\Sigma = (S, R)$ be a set system, i.e., $S$ is a universe of $n$ elements and $R \subseteq 2^S$ where $|R| = m$. A coloring is a function $\chi : S \rightarrow \{-1, 1\}$, and the discrepancy of a coloring is defined $\text{disc}(\Sigma, \chi) = \max_{R \subseteq R} \left| \sum_{r \in R} \chi(r) \right|$. The discrepancy of a set system is the discrepancy of the best coloring, i.e., $\text{disc}(\Sigma) = \min_{\chi} \text{disc}(\Sigma, \chi)$.

Show that for any $\Sigma$, there exists a coloring with $O(\sqrt{n \ln m})$ discrepancy. **Hint:** Use a random coloring and the following Chernoff bound variant: if $X_1, \ldots, X_n$ are independent random variables where $\Pr[X_i = 1] = \Pr[X_i = -1] = 1/2$, then for $\alpha > 0$, 

$$\Pr \left[ \left| \sum_{i=1}^n X_i \right| > \alpha \right] \leq 2e^{-\frac{\alpha^2}{2n}}.$$  

**Problem 5:** Recall the LP relaxation of the set cover problem. Suppose we use the following randomized rounding algorithm: first let $A = \emptyset$ and $x^*$ be the optimal LP solution. Repeat the following $2 \ln n$ times: add each set $S_i$ to $A$ with probability $x^*_i$. Output the sets in $A$ as the set cover solution.

(i) Show that the expected cost of the rounded solution after the first iteration is the optimal LP cost, $OPT^*$. Show that the probability that the final cost of $A$ is more than $(8 \log n + 4)OPT^*$ is at most $1/4$.

(ii) Show that the probability that there exists an uncovered element is at most $1/4$.

(iii) Show that the above randomized algorithm achieves an $O(\log n)$-approximation with probability at least $1/2$. 

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**Assignment 4**  
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