CompSci 102 Discrete Mathematics for CS
Spring 2006 Forbes HW 2

Due Wednesday, Feburary 15

1. (8 pts.) **L**et A, B, and C be sets. Show that

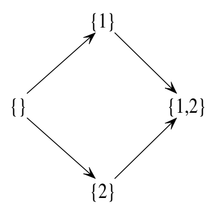
$$(A - B) - C = (A - C) - (B - C)\ .$$

[Hint: show that each side of the equation is contained in the other (a “mutual inclusion” proof).]

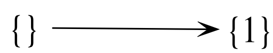
2. (8 pts.) Primes & Squares
Let A be the set of prime numbers smaller than 15, and B the set of perfect squares less than 15.

(a) How many elements are in the power set of the Cartesian product of A and B?
(b) How many elements are in the Cartesian product of the power sets of A and B?
(c) How many elements are in the power set of the power set of B?
(d) How many elements are in the power set of the power set of A?

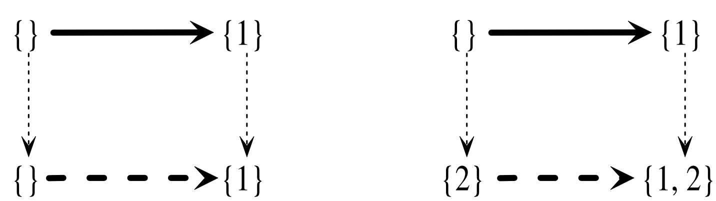
3. (28 pts.) Lattices
The power set of $S_2 = \{1, 2\}$ is $P(S) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$. Let us write the four elements of $P(S_2)$ on the corners of a square, and draw an arrow from one corner to another whenever the set at the first corner is a subset of the set at the other corner, and the first corner has exactly one fewer item than the second:



Note that there is no arrow from the empty set $\{\}$ to $\{1, 2\}$. This drawing is called a *lattice*. The lattice for the power set $P(S_1)$ of $S_1 = \{1\}$ is obviously simpler:



Interestingly, you can make the lattice for $P(S_2)$ by combining two copies of the lattice for $P(S_1)$ as follows. You first draw the two copies, and draw arrows from every corner in the first copy to its corresponding corner (that is, the corner with the same set) in the second copy. On the left side of the figure below, copy 1 is drawn with a thick, solid arrow, copy 2 with a thick, dashed arrow, so we can distinguish them, and the connections between corresponding corners are drawn with thin, dotted arrows. Finally, we go through all the sets in copy 2, and add the member of S_2 that was missing in S_1, that, is the number 2. This is shown on the right side of the figure below.



The figure obtained in this way may not look visually like the lattice for S_2 we drew earlier, but it is the same in the sense that the arrows connect the same sets in the same way.

(a) Draw the lattice for the power set $P(S_3)$ of the set S_3 without following the procedure above. Just connect the sets in the figure below with an arrow whenever one set is a subset of the other, and the first set has exactly one fewer item than the second. The arrow points from the smaller set to the bigger one. Make sure you have *all* the arrows, and that you do not draw any that do not belong.

	{1}	{1,2}	
{}	{2}	{1,3}	{1,2,3}
	{3}	{2,3}	

(b) Now redraw the lattice for the power set $P(S_3)$ of the set $S_3 = \{1, 2, 3\}$ by combining two copies of the lattice for $P(S_2)$ with the procedure used above for making $P(S_2)$ out of $P(S_1)$ (the new element in S_3 is of course the number 3). When doing so, draw arrows in the first copy with thick, solid lines, arrows in the second copy with thick, dashed lines, and arrows that connect the two copies with thin, dotted lines.

(c) The lattices you drew in your answers to questions (a) and (b) of this problem probably look visually different, but ought to be the same in the sense that the arrows connect the same sets in the same way in the two lattices. To check this, redraw the lattice in your answer to part (a) of this problem without changing the layout (*i.e.*, the relative position of the sets on paper), but using the arrow styles you used in part (b): thick, solid arrows when you connect two sets that do not contain the number 3; thick, dashed arrows when you connect two sets that both contain the number 3; thin, dotted arrows when you connect a set without the number 3 to a set with the number 3. The equivalence of the two lattices should be more obvious now.

(d) Suppose that instead of drawing your lattice for $P(S_3)$ on the plane you draw if in space. Specifically, a set $S \in P(S_3)$ is drawn at coordinates (x, y, z) according to the following rules:

$$x = \begin{cases} 1 & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S \end{cases}$$

$$y = \begin{cases} 1 & \text{if } 2 \in S \\ 0 & \text{if } 2 \notin S \end{cases}$$

$$z = \begin{cases} 1 & \text{if } 3 \in S \\ 0 & \text{if } 3 \notin S \end{cases} .$$

What geometric figure do you obtain when you draw your lattice? [Hint: draw it.]

4. (16 pts.) Commutativity and Identity

(a) A binary operator $f(x, y)$ is commutative if the result does not depend on the order of the arguments. State using quantifier notation the proposition that f is *not* commutative. (Matrix multiplication is an example of a noncommutative operator.)

(b) An *identity* for a binary operator is a element that, when combined with any element x, yields x itself. For example, the identity for $+$ is 0; the identity for $\times$ is 1. For a *noncommutative* operator, a *left-identity* is an identity when it is the first argument; a *right-identity* is an identity when it is the second argument. State using quantifier notation the propositions that i_L is a left-identity for f and that i_R is a right-identity for f.

(c) Prove that for any i_L and i_R satisfying your definitions, it must be the case that $i_L = i_R$.

(d) Have you proved that f has exactly one identity element? If not, what have you proved?

5. (24 pts.) Prove or disprove:

(a) Every positive integer can be expressed as the sum of two squares.

(b) Every positive integer can be expressed as the sum of three squares.

(c) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real x.

(d) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real x, y.

(e) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all real $x > 0$.

(f) For all rational numbers a and b, a^b is also rational.

6. (6 pts.) Modular arithmetic
Let m be a positive integer, and let a, b, and c be integers. Show that if $a \equiv b(\mathrm{mod} m)$, then $a - c \equiv b - c(\mathrm{mod} m)$

7. (10 pts.) Book Problems

(a) §2.4, Exercise 20

(b) §2.4, Exercise 60

8. (max 5 pts.) Extra credit
Let a and b be positive irational numbers such that $1/a + 1/b = 1$. Show that, for every positive integer n, there is some positive integer k such that either $n = \lfloor ka \rfloor$ or $n = \lfloor kb \rfloor$.