CompSci 102 Discrete Mathematics for CS
Spring 2006 Forbes HW 3

Due Monday, Feburary 27

Please list any resources (books, sites on the web, etc.) that you use in completing this assignment.

1. **(15 pts.) Induction**
   Prove that

   $$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

   for $n = 0, 1, \ldots$ by induciton. [Hint: when manipulating algebraic expressions, think what your goal is. This may make it more obvious how to proceed.]

   The sequence $s_n = n^2$ for $n = 0, 1, \ldots$ (the sum of this series appears in the previous problem) can be defined recursively, that is, by giving its first value and a way to compute s_{n+1} from s_n for $n = 0, 1, \ldots$.

   (a) Define s_n recursively. [Hint: there is more than one way to do this.]

   (b) Let

   $$S_n = \sum_{i=0}^{n} s_i$$

   where s_i is the same sequence as before, that is, $s_i = i^2$. Define S_n recursively. [Hint: this is trivial.]

2. **(10 pts.) Induction**
   Prove by induction that

   $$\sum_{i=0}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

   for $n = 0, 1, \ldots$.

3. (20 pts.) Fixing proofs

From D. Knuth, *The Art of Computer Programming*, (1) 1.2.1, Exercise 2.

What is wrong with the following proof? Explain carefully.

"Theorem:" For all positive real numbers x and for all positive integers n we have $x^{n-1} = 1$.

"Proof:" Let $P(n)$ be the statement $x^{n-1} = 1$. The base case $P(1)$ is true: If $n = 1$, then $x^{n-1} = x^{1-1} = x^0 = 1$. We now prove the implication $P(1), \ldots, P(k) \rightarrow P(k+1)$. We have

$$x^{(k+1)-1} = x^k = \frac{x^{k-1}\,x^{k-1}}{x^{k-2}}\ .$$

If we assume that $P(1), \ldots, P(k)$ are true, the last expression equals

$$\frac{1 \cdot 1}{1} = 1$$

by $P(k)$ and $P(k-1)$, so $P(k+1)$ holds as well. By the principle of strong induction, $P(n)$ is true for all $n = 1, 2, \ldots$.

4. (10 pts.) Recursive programs

To produce formatted output, it is often useful to compute the number of digits in the printed representation of an integer. For positive integers n, a simple formula is $1 + \lfloor \log_{10} n \rfloor$, but this is computational overkill. Prove that the following recursive program computes the same thing:

```
int digits(int n)
{
  if (n < 10)
    return 1;
  return 1 + digits(n/10);
}
```

5. (15 pts.) Induction over strings

(a) Write a (mathematical) recursive definition of the string concatenation operation $x \cdot y$ in terms of the *head* and *tail* functions. (Hint: the recursion is on x only.)

(b) Using induction over strings, prove that string concatenation is associative, i.e.,

$$\forall x, y, z\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

6. (10 pts.) Strong induction

When putting together a jigsaw puzzle, a *block* is any number of pieces joined together (including 1), and a *move* joins together two blocks of any size. Prove that it takes exactly $n-1$ moves to put together a jigsaw puzzle of n pieces *no matter what moves are done in what order*, provided no moves are ever undone.

7. (20 pts.) Tilings

(a) Prove that an 8×8 region with two opposite corner squares removed cannot be tiled using 2×1 tiles.

(b) Prove that a 10×10 region cannot be tiled using the tile shown in Figure 1(a).

(c) Prove that a 10×10 region cannot be tiled using the tile shown in Figure 1(b).

Figure 1: (a) The T 4-tile. (b) The big-L 4-tile.