Example

\( L = \{a^n ba^n \mid n > 0\} \)

Closure Properties

A set is closed over an operation if

\[
L_1, L_2 \in \text{class} \\
L_1 \text{ op } L_2 = L_3 \\
\Rightarrow L_3 \in \text{class}
\]

Example

\( L_1 = \{ x \mid x \text{ is a positive even integer}\} \)

L is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\bar{L}_1 \\
L_1^*
\]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure

complementation:
$L_1$ is reg. lang.
$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$
Construct $M'$ s.t.
final states in $M$ are
nonfinal states in $M'$
nonfinal states in $M$ are
final states in $M'$
$\Rightarrow$ closed under complementation

intersection:
$L_1$ and $L_2$ are reg. lang.
$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.
$L_1 = L(M_1)$ and $L_2 = L(M_2)$
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
$Q' = (Q \times P)$
$\delta'$:
$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$
$\Rightarrow$ closed under intersection
Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$

**Right quotient**

Def: $L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{ a^* b^* \cup b^* a^* \}$
$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$
$L_1 / L_2 = $

**Theorem** If $L_1$ and $L_2$ are regular, then $L_1 / L_2$ is regular.

**Proof** (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do
  Make $i$ the start state (representing $L'_i$)
  if $L'_i \cap L_2 \neq \emptyset$ then
    put $q_i$ in $F'$ in $M'$

QED.
**Homomorphism**

Def. Let Σ, Γ be alphabets. A homomorphism is a function

\[ h : \Sigma \rightarrow \Gamma^* \]

**Example:**

\[ \Sigma = \{a, b, c\}, \Gamma = \{0, 1\} \]

\[ h(a) = 11 \]

\[ h(b) = 00 \]

\[ h(c) = 0 \]

\[ h(bc) = \]

\[ h(ab^*) = \]

**Questions about regular languages:**

L is a regular language.

- Given L, Σ, w ∈ Σ*, is w ∈ L?

- Is L empty?

- Is L infinite?

- Does \( L_1 = L_2 \)?
Ch. 4.3 - **Identifying Nonregular Languages**

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = a^* b^*$
- $L_2 = \{a^n b^n | n > 0\}$

**Prove that** $L_2 = \{a^n b^n | n > 0\}$ **is** ?

- Proof:
**Pumping Lemma:** Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
xy^iz \in L &\text{ for all } i \geq 0
\end{align*}
\]

**Meaning:** Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in $L$.

**To Use the Pumping Lemma to prove $L$ is not regular:**

- **Proof by Contradiction.**
  Assume $L$ is regular.
  $\Rightarrow L$ satisfies the pumping lemma.
  Choose a long string $w$ in $L$, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L$ $\forall$ $i \geq 0$.
  The pumping lemma does not hold. Contradiction!
  $\Rightarrow L$ is not regular. QED.

**Example** $L=\{a^ncb^n|n > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$ where $m$ is the constant in the pumping lemma. (Note that $w$ must be chosen such that $|w| \geq m$.)
  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with $cb^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

It should be true that $xy^iz \in L$ for all $i \geq 0$. 
Example $L=\{a^n b^{n+s} c^s | n, s > 0 \}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w = 

  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with the rest of the string $b^{m+s} c^s$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

\[ \Sigma = \{a, b\}, L=\{w \in \Sigma^* \mid n_a(w) > n_b(w)\} \]

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w = 

  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

• Proof:
  Assume $L$ is regular. ⇒ the pumping lemma holds.
  Choose $w = a^3b^m c^{m-3}$ where $m$ is the constant in the pumping lemma. There are three ways to partition $w$ into three parts, $w = xyz$. 1) $y$ contains only $a$’s 2) $y$ contains only $b$’s and 3) $y$ contains $a$’s and $b$’s
  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide $w$ into three parts s.t. the pumping lemma contraints were true).
  Case 1: ($y$ contains only $a$’s). Then $x$ contains 0 to 2 $a$’s, $y$ contains 1 to 3 $a$’s, and $z$ contains 0 to 2 $a$’s concatenated with the rest of the string $b^m c^{m-3}$, such that there are exactly 3 $a$’s. So the partition is:
  
  $x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}$

  where $k \geq 0$, $j > 0$, and $k + j \leq 3$ for some constants $k$ and $j$.
  It should be true that $xy^iz \in L$ for all $i \geq 0$.
  $xy^2z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j-k} b^m c^{m-3}) = a^{3+j} b^m c^{m-3} \notin L$ since $j > 0$, there are too many $a$’s. Contradiction!
  Case 2: ($y$ contains only $b$’s) Then $x$ contains 3 $a$’s followed by 0 or more $b$’s, $y$ contains 1 to $m - 3$ $b$’s, and $z$ contains 3 to $m - 3$ $b$’s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $x = a^3 b^k \quad y = b^j \quad z = b^{m-k-j} c^{m-3}$

  where $k \geq 0$, $j > 0$, and $k + j \leq m - 3$ for some constants $k$ and $j$.
  It should be true that $xy^iz \in L$ for all $i \geq 0$.
  $xy^2z = a^3 b^{m-j} c^{m-3} \notin L$ since $j > 0$, there are too few $b$’s. Contradiction!
  Case 3: ($y$ contains $a$’s and $b$’s) Then $x$ contains 0 to 2 $a$’s, $y$ contains 1 to 3 $a$’s, and 1 to $m - 3$ $b$’s, $z$ contains 3 to $m - 1$ $b$’s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j} c^{m-3}$

  where $3 \geq k > 0$, and $m - 3 \geq j > 0$ for some constants $k$ and $j$.
  It should be true that $xy^iz \in L$ for all $i \geq 0$.
  $xy^2z = a^3 b^j a^k b^m c^{m-3} \notin L$ since $j, k > 0$, there are $b$’s before $a$’s. Contradiction!
  ⇒ There is no partition of $w$.
  ⇒ $L$ is not regular!. QED.
To Use Closure Properties to prove $L$ is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

• Proof Outline:
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  closure properties $\Rightarrow L'$ is regular.
  Contradiction!
  $L$ is not regular. QED.

Example $L=\{a^{3n}b^n c^{n-3} | n > 3\}$

$L$ is not regular.

• Proof: (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a  \quad h(b) = a  \quad h(c) = b$
  $h(L) =$
**Example** $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof**: (proof by contradiction)
  
  Assume $L$ is regular.

Example: $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.

- **Proof**:
  
  Assume $L_1$ is regular.
  
  Goal is to try to construct $\{a^n b^n | n > 0\}$ which we know is not regular.
  
  Let $L_2 = \{a^n\}$. $L_2$ is regular.
  
  By closure under right quotient, $L_3 = L_1 \setminus L_2 = \{a^n b^p a^p | 0 \leq p \leq n, n > 0\}$ is regular.
  
  By closure under intersection, $L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n | n > 0\}$ is regular.
  
  Contradiction, already proved $L_4$ is not regular!
  
  Thus, $L_1$ is not regular. QED.