Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\( L_1 = \{ x \mid x \text{ is a positive even integer} \} \)

\( L \) is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[
\begin{align*}
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\overline{L}_1 \\
L_1^* \\
\end{align*}
\]

are regular languages.
Proof(sketsh)

$L_1$ and $L_2$ are regular languages
⇒ ∃ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
⇒ closed under union

$r_1r_2$ is r.e. denoting $L_1L_2$
⇒ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$
⇒ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- final states in $M$ are nonfinal states in $M'$
- nonfinal states in $M$ are final states in $M'$

$\Rightarrow$ closed under complementation
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:
Regular languages are closed under

reversal \( L^R \)
difference \( L_1 - L_2 \)
right quotient \( L_1 / L_2 \)
homomorphism \( h(L) \)
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
L_1 = \{ a^*b^* \cup b^*a^* \}
\]
\[
L_2 = \{ b^n \mid n \text{ is even, } n > 0 \}
\]
\[
L_1/L_2 =
\]
Theorem If \( L_1 \) and \( L_2 \) are regular, then \( L_1/L_2 \) is regular.

Proof (sketch)

\[ \exists \text{DFA } M=(Q, \Sigma, \delta, q_0, F) \text{ s.t. } L_1 = L(M). \]

Construct DFA \( M'=(Q, \Sigma, \delta, q_0, F') \)

For each state \( i \) do

- Make \( i \) the start state (representing \( L'_i \))
- if \( L'_i \cap L_2 \neq \emptyset \) then
  - put \( q_i \) in \( F' \) in \( M' \)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$ h: \Sigma \rightarrow \Gamma^* $$

Example:

$$ \Sigma = \{a, b, c\}, \Gamma = \{0, 1\} $$

$$ h(a) = 11 $$
$$ h(b) = 00 $$
$$ h(c) = 0 $$

$$ h(bc) = $$

$$ h(ab^*) = $$
Questions about regular languages:
L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?

• Is L empty?

• Is L infinite?

• Does L₁ = L₂?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = aa^*bb^*$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that \( L_2 = \{a^n b^n | n > 0\} \) is ?

- Proof:
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \text{ for all } i \geq 0
\]
To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.
  Assume L is regular.
  ⇒ L satisfies the pumping lemma.
  Choose a long string \( w \) in L,
  \( |w| \geq m \).
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m \), \( |y| \geq 1 \) and \( xy^i z \in L \ \forall \ i \geq 0 \).

The pumping lemma does not hold. Contradiction!
⇒ L is not regular. QED.
Example \( L = \{ a^n c b^n \mid n > 0 \} \)

\( L \) is not regular.

• Proof:
  Assume \( L \) is regular.
  \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = \)
Example \( L = \{a^n b^{n+s} c^s | n, s > 0 \} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = \)
  
  So the partition is:
Example $\Sigma = \{a, b\}$, 
$L = \{w \in \Sigma^* | n_a(w) > n_b(w)\}$

$L$ is not regular.

• Proof:
  Assume $L$ is regular.
  ⇒ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} \mid n > 3\}$

$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- Proof Outline:
  
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular.
  Contradiction!
  
  $L$ is not regular. QED.
Example \( L = \{ a^3 b^n c^{n-3} | n > 3 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  
  Assume \( L \) is regular.
  
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  
  \[
  h(a) = a \quad h(b) = a \quad h(c) = b
  \]
  
  \( h(L) = \)
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof: (proof by contradiction)**
  Assume $L$ is regular.
Example: $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.