Fourth Homework Assignment

Write the solution to each problem on a single page. The deadline for handing in solutions is 18 March 2009.

Question 1. ($20 = 10 + 10$ points).

(a) Prove the following claim:

$$1 + 7 + \cdots + (3n^2 - 3n + 1) = n^3.$$

(b) (Problem 4.1-11 in our textbook). Find the error in the following proof that all positive integers n are equal. Let $p(n)$ be the statement that all numbers in an n-element set of positive integers are equal. Then $p(1)$ is true. Let $n \geq 2$ and write N for the set of n first positive integers. Let N' and N'' be the sets of $n-1$ first and $n-1$ last integers in N. By $p(n-1)$, all members of N' are equal, and all members of N'' are equal. Thus, the first $n-1$ elements of N are equal and the last $n-1$ elements of N are equal, and so all elements of N are equal. Therefore, all positive integers are equal.

Question 2. (20 points). Recall the Chinese Remainder Theorem stated for two positive, relatively prime moduli, m and n, in Section 7. Assuming this theorem, prove the following generalization by induction on k.

CLAIM. Let $n_1, n_2, \ldots, n_k$ be positive, pairwise relative prime numbers. Then for every sequence of integers $a_i \in \mathbb{Z}_{n_i}$, $1 \leq i \leq k$, the system of k linear equations,

$$x \bmod n_i \quad = \quad a_i,$$

has a unique solution in $\mathbb{Z}_N$, where $N = \prod_{i=1}^{k} n_i$.

Question 3. ($20 = 10 + 10$ points).

(a) (Problem 4.2-13 in our textbook). Solve the recurrence $T(n) = 2T(n-1) + 3^n$, with $T(0) = 1$.

(b) (Problem 4.2-17 in our textbook). Solve the recurrence $T(n) = rT(n-1) + n$, with $T(0) = 1$. (Assume that $r \neq 1$.)

Question 4. ($20 = 7 + 7 + 6$ points). Consider the following algorithm segment.

```
int FUNCTION(n)
  if n > 0 then
    n = FUNCTION(⌊n/a⌋) + FUNCTION(⌊n/b⌋)
  endif
  return n.
```

We can assume that $a, b > 1$, so the algorithm terminates. In the following questions, let a_n be the number of iterations of the `while` loop.

(a) Find a recurrence relation for a_n.

(b) Find an explicit formula for a_n.

(c) How fast does n grow? (big Θ terms)

Question 5. ($20 = 4{+}4{+}4{+}4{+}4$ points). (Problem 4.4-1 in our textbook). Use the Master Theorem to solve the following recurrence relations. For each, assume $T(1) = 1$ and n is a power of the appropriate integer.

(a) $T(n) = 8T(\frac{n}{2}) + n$.

(b) $T(n) = 8T(\frac{n}{2}) + n^3$.

(c) $T(n) = 3T(\frac{n}{2}) + n$.

(d) $T(n) = T(\frac{n}{4}) + 1$.

(e) $T(n) = 3T(\frac{n}{3}) + n^2$.