2 Binomial Coefficients

In this section, we focus on counting the number of ways sets and lists can be chosen from a given set.

Permutations. A *permutation* is a bijection from a finite set D to itself, $f : D \to D$. For example, the permutations of $\{1, 2, 3\}$ are: $123, 132, 213, 231, 312$, and 321. Here we list the permutations in lexicographic order, same as they would appear in a dictionary. Assuming $|D| = k$, there are $k!$ permutations or, equivalently, orderings of the set. To see this, we note that there are k choices for the first element, $k - 1$ choices for the second, $k - 2$ for the third, and so on. The total number of choices is therefore $k(k-1) \cdot \ldots \cdot 1$, which is the definition of $k!$.

Let $N = \{1, 2, \ldots, n\}$. For $k \le n$, a *k-element permutation* is an injection $\{1, 2, \ldots, k\} \to N$. In other words, a k-element permutation is a list of k distinct elements from N. For example, the 3-element permutations of $\{1, 2, 3, 4\}$ are

123,	124,	132,	134,	142,	143,
213,	214,	231,	234,	241,	243,
312,	314,	321,	324,	341,	342,
412,	413,	421,	423,	431,	432.

There are 24 permutations in this list. There are six orderings of the subset $\{1, 2, 3\}$ in this list. In fact, each 3-element subset occurs six times. In general, we write $n^{\underline{k}}$ for the number of k-element permutations of a set of size n. We have

$$\begin{aligned} n^{\underline{k}} &= \prod_{i=0}^{k-1} (n - i) \\ &= n(n-1) \cdots (n - (k-1)) \\ &= \frac{n!}{(n-k)!}. \end{aligned}$$

Subsets. The *binomial coefficient* $\binom{n}{k}$, pronounced n choose k, is by definition the number of k-element subsets of a size n set. Since there are $k!$ ways to order a set of size k, we know that $n^{\underline{k}} = \binom{n}{k} \cdot k!$ which implies

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

We fill out the following tables with values of $\binom{n}{k}$, where the row index is the values of n and the column index is the value of k. Values of $\binom{n}{k}$ for $k > n$ are all zero and are omitted from the table.

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

By studying this table, we notice several patterns.

- $\binom{n}{0} = 1$. In words, there is exactly one way to choose no item from a list of n items.
- $\binom{n}{n} = 1$. In words, there is exactly one way to choose all n items from a list of n items.
- Each row is symmetric, that is, $\binom{n}{k} = \binom{n}{n-k}$.

This table is also known as Pascal's Triangle. If we draw it symmetric between left and right then we see that each entry in the triangle is the sum of the two entries above it in the previous row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's Relation. We express the above recipe of constructing an entry as the sum of two previous entries more formally. For convenience, we define $\binom{n}{k} = 0$ whenever $k < 0$, $n < 0$, or $n < k$.

PASCAL'S RELATION. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

PROOF. We give two arguments for this identity. The first works by algebraic manipulations. We get

$$\begin{aligned} \binom{n}{k} &= \frac{(n-k)(n-1)! + k(n-1)!}{(n-k)!k!} \\ &= \frac{(n-1)!}{(n-k-1)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!} \\ &= \binom{n-1}{k} + \binom{n-1}{k-1}. \end{aligned}$$

For the second argument, we partition the sets. Let $|S| = n$ and let a be an arbitrary but fixed element from S. $\binom{n}{k}$ counts the number of k-element subsets of S. To get the number of subsets that contain a, we count the $(k-1)$-element subsets of $S - \{a\}$, and to get the number of subsets that do not contain a, we count the k-element subsets of $S - \{a\}$. The former is $\binom{n-1}{k-1}$ and the latter is $\binom{n-1}{k}$. Since the subsets that contain a are different from the subsets that do not contain a, we can use the Sum Principle 1 to get the number of k-element subsets of S equal to $\binom{n-1}{k-1} + \binom{n-1}{k}$, as required. ◻

Binomials. We use binomial coefficients to find a formula for $(x+y)^n$. First, let us look at an example.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y) \\ &= xx + yx + xy + yy \\ &= x^2 + 2xy + y^2.\end{aligned}$$

Notice that the coefficients in the last line are the same as in the second line of Pascal's Triangle. This is more generally the case and known as the

BINOMIAL THEOREM. $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$.

PROOF. If we write each term of the result before combining like terms, we list every possible way to select one x or one y from each factor. Thus, the coefficient of $x^{n-i}y^i$ is equal to $\binom{n}{n-i} = \binom{n}{i}$. In words, it is the number of ways we can select $n-i$ factors to be x and have the remaining i factors to be y. This is equivalent to selecting i factors to be y and have the remaining factors be x. ◻

Corollaries. The Binomial Theorem can be used to derive a number of other interesting sums. We prove three such consequences.

COROLLARY 1. $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

PROOF. Let $x = y = 1$. Then, by the Binomial Theorem we have

$$(1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i.$$

This implies the claimed identity. ◻

COROLLARY 2. $\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$.

PROOF. We use Pascal's Relation to prove this identity. It is instructive to trace our steps graphically, in the triangle above. In a first step, we replace $\binom{n+1}{k+1}$ by $\binom{n}{k}$ and $\binom{n}{k+1}$. Keeping the first term, we replace the second, $\binom{n}{k+1}$, by $\binom{n-1}{k}$ and $\binom{n-1}{k+1}$. Repeating this operation, we finally replace $\binom{k+1}{k+1}$ by $\binom{k}{k} = 1$ and $\binom{k}{k+1} = 0$. In other words, $\binom{n+1}{k+1}$ is equal to the sum of the $\binom{j}{k}$ for j running from n down to k. ◻

COROLLARY 3. $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

PROOF. We first express the summands in terms of binomial coefficients and then use Corollary 2 to get the result.

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= 2\sum_{i=1}^{n} \frac{i^2 - i}{2} + \sum_{i=1}^{n} i \\ &= 2\sum_{i=1}^{n} \binom{i}{2} + \sum_{i=1}^{n} \binom{i}{1} \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= \frac{2(n+1)n(n-1)}{1 \cdot 2 \cdot 3} + \frac{(n+1)n}{1 \cdot 2} \\ &= \frac{n^3 - n}{3} + \frac{n^2 + n}{2}.\end{aligned}$$

This implies the claimed identity. ◻

Summary. The binomial coefficient, $\binom{n}{k}$, counts the different ways we can choose k elements from a set of n. We saw how it can be used to compute $(x+y)^n$. We proved several corollaries and saw that describing the identities as counting problems can lead us to different, sometimes simpler proofs.