What is **game theory**?

- Game theory studies settings where multiple parties (agents) each have
  - different preferences (utility functions),
  - different actions that they can take
- Each agent’s utility (potentially) depends on all agents’ actions
  - What is optimal for one agent depends on what other agents do
    - Very circular!
- Game theory studies how agents can rationally form beliefs over what other agents will do, and (hence) how agents should act
  - Useful for acting as well as (potentially) predicting behavior of others
- Game theory does not directly aim to be a descriptive theory
Real World Game Theory Examples

• War
• Auctions
• Animal behavior
• Networking protocols, peer to peer networking behavior
• Road traffic

• Mechanism design: Suppose we want people to do X? How do we engineer the situation so that they will act that way?

Penalty kick example

Is this a “rational” outcome? If not, what is?
**Rock-paper-scissors**

<table>
<thead>
<tr>
<th></th>
<th>Row player</th>
<th>Column player</th>
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<tbody>
<tr>
<td>0, 0</td>
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Row player AKA player 1 chooses a row

Column player AKA player 2 (simultaneously) chooses a column

A row or column is called an action or (pure) strategy

Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

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**“Chicken”**

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

<table>
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S D
D S

not zero-sum
Rock-paper-scissors – Seinfeld variant

MICKEY: All right, rock beats paper!
(Mickey smacks Kramer’s hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]

Dominance

• Player i’s strategy \(s_i\) strictly dominates \(s'_i\) if
  – for any \(s_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})\)

• \(s_i\) weakly dominates \(s'_i\) if
  – for any \(s_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})\); and
  – for some \(s_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})\)

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]
Prisoner’s Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

\[
\begin{array}{cc|c|c}
& 
\text{confess} & \text{don’t confess} \\
\text{confess} & -2, -2 & 0, -3 \\
\text{don’t confess} & -3, 0 & -1, -1 \\
\end{array}
\]

“Should I buy an SUV?”

\[
\begin{array}{ccc}
\text{purchasing + gas cost} & \text{accident cost} \\
\text{cost: 5} & \text{cost: 5} & \text{cost: 5} \\
\text{cost: 3} & \text{cost: 8} & \text{cost: 2} \\
\text{cost: 5} & \text{cost: 5} & \text{cost: 5} \\
\end{array}
\]

\[
\begin{array}{cc|c|c}
& 
\text{purchasing + gas cost} & \text{accident cost} \\
\text{cost: 5} & \text{cost: 5} & \text{cost: 5} \\
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\]
“2/3 of the average” game

• Everyone writes down a number between 0 and 100
• Person closest to 2/3 of the average wins
• Example:
  – A says 50
  – B says 10
  – C says 90
  – Average(50, 10, 90) = 50
  – 2/3 of average = 33.33
  – A is closest (|50-33.33| = 16.67), so A wins

Iterated dominance

• Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
• Iterated strict dominance on Seinfeld’s RPS:

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0, 0 & 1, -1 \\
-1, 1 & 0, 0 \\
\end{array}
\]
“2/3 of the average” game revisited

\[
\begin{align*}
100 & \quad \{ \text{dominated} \\
(2/3) \times 100 & \quad \{ \text{dominated after removal of (originally)} \\
(2/3) \times (2/3) \times 100 & \quad \text{dominated strategies}
\end{align*}
\]

Mixed strategies

- **Mixed strategy** for player i = probability distribution over player i’s (pure) strategies
- E.g. 1/3, 1/3, 1/3
- Example of dominance by a mixed strategy:

\[
\begin{array}{cc}
1/2 & 1/2 \\
3, 0 & 0, 0 \\
0, 0 & 3, 0 \\
1, 0 & 1, 0
\end{array}
\]
Nash equilibrium [Nash 50]

- A vector of strategies (one for each player) is called a strategy profile.
- A strategy profile \((\sigma_1, \sigma_2, ..., \sigma_n)\) is a Nash equilibrium if each \(\sigma_i\) is a best response to \(\sigma_{-i}\)
  - That is, for any \(i\), for any \(\sigma_i', u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})\)
- Note that this does not say anything about multiple agents changing their strategies at the same time.
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]

- (Note - singular: equilibrium, plural: equilibria)

Nash equilibria of “chicken”

\[
\begin{array}{c|cc}
&D & S \\
\hline
D & 0, 0 & -1, 1 \\
S & 1, -1 & -5, -5 \\
\end{array}
\]

- \((D, S)\) and \((S, D)\) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria
Rock-paper-scissors

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- Any pure-strategy Nash equilibria?
- But it has a mixed-strategy Nash equilibrium:
  Both players put probability 1/3 on each action.
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize

Nash equilibria of “chicken”...

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- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1’s utility for playing D = -p_S
- Player 1’s utility for playing S = p_D - 5p_S = 1 - 6p_S
- So we need -p_S = 1 - 6p_S which means p_S = 1/5
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player
Computational Issues

- Zero-sum games can be solved efficiently as linear programs (see slides from earlier in the semester)
- General sum games may require exponential time (in # of actions) to find a single equilibrium (non known efficient algorithm and good reasons to suspect that none exists)

- Some better news: Despite bad worst-case complexity, many games can be solved quickly

Game Theory Issues

- How descriptive is game theory?
  - Some evidence that people play equilibria
  - Some evidence that people act irrationally
  - If it is computationally intractable to solve for equilibria of large games, it would seem unlikely that people are doing this

- How reasonable is game theory?
  - Are payoffs known?
  - Are situations really simultaneous move with no information about how the other player will act?
  - Are situations really single-shot
Extensions

• Partial information (just as MDPs are extended to POMDPs)
• Uncertainty about the game parameters, e.g., payoffs (Bayesian games)
• Multistep games with distributions over next states (game theory + MDPs = stochastic games)
• Multistep + partial information (Partially observable stochastic games)

• Game theory is so general, that it can encompass essentially all aspects of strategic, multiagent behavior, e.g., negotiating, threats, bluffs, coalitions, bribes, etc.