Optimal and Robust Mechanism Design with Interdependent Values

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We study interdependent value settings [Milgrom and Weber 1982] and extend several fundamental results from the well-studied independent private values model to these settings. For revenue-optimal mechanism design, we give conditions under which Myerson's virtual value-based mechanism remains optimal with interdependent values. One of these conditions is robustness of the truthfulness and individual rationality guarantees, in the sense that they are required to hold ex post. We then consider an even more robust class of mechanisms called "prior independent" (a.k.a. "detail free"), and show that by simply using one of the bidders to set a reserve price, it is possible to extract near-optimal revenue in an interdependent values setting. This shows that a considerable level of robustness is achievable for interdependent values in single-parameter environments.

Categories and Subject Descriptors: [**Theory of Computation**]: General

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: interdependence; correlated values; optimal auctions; Myerson theory; prior-independence

1. INTRODUCTION

The subject of this paper is optimal and robust mechanism design in the classic model of interdependent values introduced by Milgrom and Weber [1982]. The model of interdependent values is not only of economic importance in itself, but also sheds new light on the inherent tradeoff between revenue maximization and robustness in the design of mechanisms.

In technical terms, we study optimal and approximately-optimal mechanisms in single-parameter settings, with robust guarantees of ex post incentive compatibility and individual rationality; the approximately-optimal mechanisms we design are also prior-independent, that is, robust to distributional details.

1.1. IPV versus Interdependent Values

Economic research on auctions has explored different valuation models over the past decades, which can roughly be divided into *independent private*[1] values (*IPV*), versus

[1]The term "private" is used here as opposed to "common", not to be confused with private in the sense of "non-public", as in the context of information asymmetries. In what follows to avoid confusion we shall refer to non-public information as "privately-known".

This research was supported in part by NSF Awards CCF-1016885 and CCF-1215965 and an ONR PECASE Award. The second author is supported by the Hsieh Family Stanford Interdisciplinary Graduate Fellowship.
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An earlier version of this work appeared in the 14th ACM Conference on Economics and Computation (EC'13; formerly known as ACM Conference on Electronic Commerce), 2013.

DOI 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

the more general class of *interdependent* values that allows for correlation and commonality (see [Krishna 2010, Chapters 2-5; 13-17] versus [Krishna 2010, Chapters 6-10; 18]). The more nacent research effort in theoretical computer science has focused largely on the more restricted IPV model, recently also venturing into the realm of correlation (see, notably, [Papadimitriou and Pierrakos 2011; Dobzinski et al. 2011; Cai et al. 2012]). A broad research goal is therefore to apply the computer science lens to the study of mechanisms for the general interdependent model. This paper takes first steps in that direction by unifying and generalizing previous results to establish the necessary technical foundation, and demonstrates that there are natural sufficient conditions under which positive results in the form of mechanisms with very strong guarantees can be achieved.

The importance of the interdependent model in the economic literature stems from the fact that for many high-stake auctions that arise in practice, interdependent values are a more realistic model of bidders' values than IPV. Interdependence captures situations in which every bidder has only partial information, called his *signal*, about his value for winning the auction, and this information can be correlated with other bidders' information. Furthermore, the information held by other bidders may *directly* affect the bidder's value – mathematically his value is a function of his own signal and the (possibly correlated) signals of his competitors.

A classic example from the economic literature is the *mineral rights model* [Wilson 1969]. In auctions for oil drilling, the value of the drilling rights is determined by whether or not there is oil to be found on the drilling site. This value is often common – yet unknown – to all bidders. However, typically every bidder has some private noisy signal regarding the value, achieved by, for example, conducting a geological survey. Not only are these signals positively correlated, but the information gathered by the other bidders would certainly change a bidder's expected value for winning if he gained access to it.

Note that the IPV model is not rich enough to capture the described informational setting: In the IPV model an attempted approximation is that values all come from a distribution over a high-valued support (oil exists), or a low-valued one (no oil). But to model that the seller does not know which of the two supports is the case will make the bidders correlated in his view. Similarly, we need interdependence to model that bidders do not know their precise value for winning an auction since it depends on others' information. The model of interdependence thus enriches the set of underlying assumptions we are able to make about the informational structure of the auction setting. In this paper we explicitly treat such informational assumptions and their role in designing mechanisms, as discussed next.

1.2. Robustness in Mechanism Design

A second goal of this paper is the design of *robust* mechanisms. Consider the informational assumptions of a standard Bayesian auction environment, where bidders have privately-known information.

Informational assumption 1. The bidders all know the probability distribution of the privately-known information, and make strategic decisions accordingly.

Informational assumption 2. The seller knows the probability distribution of the privately-known information, and chooses the mechanism (or sets parameters such as reserve prices) accordingly.

There are many theoretical and practical reasons to wish to relax the above assumptions; for example, accurate *prior* information may be expensive to acquire, or a mechanism may need to be re-used in settings with different distributions. In fact,

Table I. Terminology. The term *interdependent values* encompasses all four value classes described in the table.

	Private	Non-private (common)
Independent	Independent private values (IPV)	Non-private values
Correlated	Correlated values	Correlated non-private values

the issue of robustness to informational assumptions as above is "an old theme in the mechanism design literature" [Bergemann and Morris 2005] (see further discussion in Section 3.2 below). However, potentially there can be a trade-off between robustness and the objectives of the mechanism, in our case maximizing revenue.

Remarkably, Myerson [1981] showed that in the IPV model there is no trade-off with respect to the first assumption:[2] The optimal mechanism among all mechanisms that make assumption 1 does not actually use this assumption in any way. More formally, Myerson's optimal mechanism is *ex post* incentive compatibile (*IC*) and individually rational (*IR*), but is optimal among all *Bayesian* IC and *interim* IR mechanisms (see Section 3.1 for definitions of these solution concepts).

On the other hand, in the context of interdependent values, the trade-off not only exists but becomes very extreme. A mechanism making assumption 1 can extract as revenue essentially the full welfare arising from the auction, leaving the bidders with virtually zero utility from participation [Crémer and McLean 1985; 1988]. Intuitively, dependencies among the bidders "cancel out" their strategic advantage from privately-held information and nullifies their information rents. Without assumption 1 however, the gap between the optimal expected revenue and full welfare can be arbitrarily large.

While relaxing assumption 1 leads to a loss of revenue, a possibly surprising property is regained. It turns out that Myerson's result [1981], showing the fundamental connection between the allocation and payment rules needed to induce truthfulness, extends to interdependent values once we restrict attention to mechanisms that are ex post. We show that under conditions well-studied in the literature, it is possible to follow the same path as in Myerson's original paper and get an analogue of the Myerson optimal mechanism for interdependent values, where optimality is among all ex post mechanisms. This relates the goal of robustness to that of expanding the theory for interdependent values – the latter is made possible by imposing robustness as a requirement.

We now revisit the second informational assumption, and ask what is achievable for interdependent values when *both* assumptions are relaxed. Even in the IPV model, the Myerson mechanism heavily depends on assumption 2. The question of designing new mechanisms without this assumption has been studied in the IPV model, leading to the theory of *prior-independence*. We give the first prior-independence result beyond the IPV model; in particular we show that while interdependence does complicate the problem of relaxing assumption 2, a considerable level of robustness is still achievable without giving up too much expected revenue.

1.3. Our Results

This paper makes two main technical contributions. To describe these and for the remainder of the paper we use the terminology in Table I.

(1) We develop a general analogue of Myerson's optimal auction theory that applies to many interdepedent settings of interest. While Myerson's theory does not hold in general for interdependent values (indeed, there are settings in which the Cremer-McLean mechanism extracts higher revenue than the Myerson mechanism), we show it is partially recovered when we impose ex post rather than Bayesian IC and IR con-

[2]For now we assume that the second assumption holds.

straints. This synthesizes and generalizes many known results in the economic literature (see related work in Section 4).

We first apply standard techniques to characterize ex post IC and IR mechanisms in the interdependent model and to show that their expected revenue equals their expected "conditional virtual surplus". Notably, we use the characterization to identify sufficient conditions under which the simple, "ironless" form of the Myerson mechanism is optimal. Under these conditions, the optimal mechanism simply allocates to the bidders with highest non-negative (conditional) virtual values.

(2) For non-private value settings we analyze a *prior-independent* auction and show that it is simultaneously near-optimal across a range of possible prior distributions. In particular, we adapt the *single sample* approach of Dhangwatnotai et al. [2010] to interdependent values, and show that with an additional sufficient and necessary MHR assumption, this approach results in an approximately-optimal, prior-independent mechanism.

Our prior-independence result demonstrates that non-trivial research questions can arise even in the simplest interdependent settings. Our Myerson-like characterization suggests that many interesting mechanism design results should be possible, even when bidders have interdependent values.

1.4. Organization

In Sections 2 and 3 we present the model and the basic ex post solution concept. In Section 4 we survey related work. Section 5 develops the first result above and the second result appears in Section 6. The study of interdependent settings raises many further research directions, several of which appear in Section 7.

2. MODEL

2.1. Interdependent Values Model

Single-parameter environments. We consider single-parameter Bayesian auction environments $(E, \mathcal{I})$, where $E = \{1, \ldots, n\}$ is a set of bidders, and $\mathcal{I} \subseteq 2^E$ is a non-empty collection of *feasible* bidder subsets, i.e., subsets of bidders who can win the auction simultaneously. $(E, \mathcal{I})$ is a downward-closed set system, in which a subset of a feasible subset is also feasible. A canonical example of a single-parameter environment is a multi-unit auction with unit-demand bidders, where $\mathcal{I}$ is all sets of bidders for which the number of bidders is at most the number of units. Our results are generally of interest even for single-item auctions.

Signals and interdependent values. The bidders have possibly correlated, privately-known *signals* $s_1, \ldots, s_n$, drawn from a joint distribution F with density f over the support $[0, \omega_i]^n$ (ω_i may be ∞). We adopt the standard assumptions that f is continuous and nowhere zero. Every bidder i has a publicly-known *valuation function* v_i whose arguments are the signals, and his *interdependent value* for winning is $v_i(\vec{s})$. Interdependent values are also called *information externalities* among the bidders. When the bidders share the same valuation function we say they have a *pure common value*. We impose the following standard assumptions on the valuation function $v_i(\cdot)$:

— Non-negative and normalized ($v_i(\vec{0}) = 0$);
— Twice continuously differentiable;
— Non-decreasing in all variables, strictly increasing in s_i.
— Finite expectation $\mathbb{E}_{\vec{s}}[v_i(\vec{s})] < \infty$.

Encompassed value models. The described interdependent values model is very general; it includes several narrower settings of interest (recall Table I):

(1) *Correlated values* settings, in which $v_i(\vec{s}) = s_i$ for every i and the signals/values are correlated;
(2) *Non-private values* settings, in which the signals are independent (drawn independently from distributions $\{F_i\}$) and the valuation functions $v_i(\vec{s})$ are general;
(3) *Independent private values (IPV)* settings, in which both $v_i(\vec{s}) = s_i$ for every i and the signals/values are independent.

Notation. Fix a signal profile s_{-i}. Let $v_{i|s_{-i}}(\cdot)$ denote bidder i's value given s_{-i} as a function of his signal s_i. Since v_i is strictly increasing in s_i, then $v_{i|s_{-i}}(\cdot)$ is invertible; denote by $v^{-1}_{i|s_{-i}}(\nu)$ or $v_i^{-1}(\nu \mid s_{-i})$ the signal s_i such that $v_{i|s_{-i}}(s_i) = \nu$. Slightly abusing notation, given s_{-i} denote the derivative of $v_{i|s_{-i}}(\cdot)$ at s_i by

$$\frac{d}{ds_i} v_i(\vec{s}) = \frac{d}{ds_i} v_{i|s_{-i}}(s_i).$$

2.2. Motivating Examples

We describe two natural and standard examples of non-private values. In the first example, bidders' values directly depend on the private preferences of the others. In the second example, bidders' values depend on a hidden stochastic "state of the world", of which others may posses private knowledge. We then give an example of correlated signals.

Example 2.1 (*Weighted-sum values*). Let $\beta \in [0, 1]$. Every bidder's value is a sum of his own signal and a weighted sum of the other signals:

$$v_i(\vec{s}) = s_i + \beta \sum_{j \neq i} s_j.$$

This is a simplified version of Myerson's value with *revision effects* [Myerson 1981], and when $\beta = 1$, this results in the *wallet game* [Klemperer 1998]. Weighted-sum values are a plausible value model for a painting sold in auction; a bidder's value for the painting is determined by his own appreciation of it, combined with the painting's "resale value" based on how much others appreciate it.

Example 2.2 (*Conditionally-independent values, a.k.a mineral rights model*). Bidders have a hidden stochastic pure common value v, modeled by a random variable V drawn from a publicly known distribution F_V. An important feature of the mineral rights model is that conditional on the event $V = v$, bidders' signals are independent. Furthermore, each signal is an unbiased estimator of V (its expectation when $V = v$ equals v). The bidders' effective value – their value for all operational purposes – is the pure common value

$$v_i(\vec{s}) = \mathbb{E}_{V \sim F_V}[V \mid \vec{s}].$$

The mineral rights model was developed to capture values in auctions for oil drilling leases [Wilson 1969]. Such values are determined by the true amount of existing oil, but uncertainty and information asymmetries regarding this amount creates interdependency.

A concrete setting of interest is the one in which F_V is distributed normally with parameters μ_V, σ_V (assume μ_V is far from 0 and σ_V is small), and the signals are $s_i = v + \eta_i$, where $\eta_1, \ldots, \eta_n$ are i.i.d. samples drawn from the normal distribution with parameters $\mu_\eta = 0$ and some small σ_η. In this case the value is a linear combination of the prior and empirical means, similar to Example 2.1 above:

$$v_i(\vec{s}) = \frac{n\sigma_V^2}{n\sigma_V^2 + \sigma_\eta^2}(\frac{1}{n}\sum_i s_i) + \frac{\sigma_\eta^2}{n\sigma_V^2 + \sigma_\eta^2}\mu_V.$$

Up to normalization, the coefficient of the empirical mean $\frac{1}{n}\sum_i s_i$ is the prior variance and the coefficient of the prior mean μ_V is the noise variance.

Example 2.3 (*Signals drawn from a multivariate normal distribution*). The signals in Example 2.1 above can be arbitrarily correlated. A concrete example of a joint signal distribution is a symmetric multivariate normal distribution. This distribution is "nice" in the sense that its marginals are normal as well, and if all pairwise covariances are nonnegative, then signals drawn from it satisfy a strong form of correlation called *affiliation*. We will make use of these properties below.

2.3. Conditional Virtual Values and Regularity

Fix a bidder i and a signal profile s_{-i}. The *conditional marginal density* $f_i(\cdot \mid s_{-i})$ of bidder i's signal given s_{-i} is

$$f_i(s_i \mid s_{-i}) \; = \; \frac{f(\vec{s})}{\int_0^{\omega_i} f(t, s_{-i})dt}.$$

We denote the corresponding distribution by $F_i(\cdot \mid s_{-i})$. The *conditional revenue curve* $B_i(\cdot \mid s_{-i})$ of bidder i is

$$B_i(s_i \mid s_{-i}) = v_i(\vec{s}) \int_{s_i}^{\omega_i} f_i(t \mid s_{-i})dt.$$

The conditional revenue curve represents the expected revenue from setting a threshold price $v_i(s_i, s_{-i})$ for bidder i given that the other signals are s_{-i}.[3] We can now define the *conditional virtual value* of bidder i as

$$\varphi_i(s_i \mid s_{-i}) = -\frac{\frac{d}{ds_i}B_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} = v_i(\vec{s}) - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} \cdot \frac{d}{ds_i}v_i(\vec{s}). \tag{1}$$

For private values, the conditional virtual value simplifies to a more familiar form:

$$\varphi_i(s_i \mid s_{-i}) = s_i - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})}, \tag{2}$$

For independent common values it simplifies to the form

$$\varphi_i(s_i \mid s_{-i}) = v_i(\vec{s}) - \frac{1 - F_i(s_i)}{f_i(s_i)} \cdot \frac{d}{ds_i}v_i(\vec{s}). \tag{3}$$

Regularity and MHR. We say that $F_i(\cdot \mid s_{-i})$ is *regular* if the conditional virtual value $\varphi_i(\cdot \mid s_{-i})$ is weakly increasing; we say it has monotone hazard rate, or *MHR*, if the inverse hazard rate $(1 - F_i(s_i \mid s_{-i}))/f_i(s_i \mid s_{-i})$ is weakly decreasing. Its *monopoly price* is the signal s_i such that the conditional virtual value $\varphi_i(s_i \mid s_{-i})$ equals zero.

Conditional vs. standard virtual values. Equation (1) reveals three complications introduced by conditional virtual values in comparison to standard virtual values in the IPV model: first, the value v_i is a function of others' signals s_{-i}; second, the inverse hazard rate is conditional on s_{-i}; and third, there is an extra term dv_i/ds_i. The

[3]Equation 2.3 uses the assumption that v_i is strictly increasing in s_i, so that the probability with which bidder i's value is at least $v_i(\vec{s})$ equals the probability $\int_{s_i}^{\omega_i} f_i(t \mid s_{-i})dt$ that bidder i's signal is at least s_i.

Myerson-like mechanisms we study below will rank bidders according to their conditional virtual values; while in the IPV model regularity is sufficient for such a mechanism to be IC, these three complications suggest that assumptions beyond regularity will be required. For example, regularity of the signal distribution restricts only the conditional marginals, whereas for a joint distribution more "global" constraints may be necessary.

2.4. Auction Settings of Interest

We present several settings of particular interest which are extensively studied in the literature. These settings arise by imposing natural further assumptions on a general single-parameter auction environment, and we will repeatedly refer to them in our results.

2.4.1. Matroid settings. Matroid settings arise by imposing a "substitutes" structure on the feasible bidder subsets. The non-empty, downward-closed system $(E, \mathcal{I})$ of bidders and feasible subsets is a *matroid* if the following *exchange property* holds: for every $S, T \in \mathcal{I}$ such that $|S| > |T|$, there is some bidder $i \in S \setminus T$ such that $T \cup \{i\} \in \mathcal{I}$ (see, e.g., [Oxley 1992]). Examples of matroid settings include digital goods where $\mathcal{I} = 2^E$, k-unit auctions where $\mathcal{I}$ is all subsets of size at most k, and certain unit-demand matching markets corresponding to transversal matroids.

2.4.2. Settings with regularity or MHR. Settings in which the signal distribution F is regular (resp., MHR). That is, for every bidder i and signal profile s_{-i}, the conditional marginal distribution $F_i(\cdot \mid s_{-i})$ is regular (resp., MHR). The multivariate normal distribution in Example 2.3 is MHR and thus also regular. Regularity arises in the IPV setting as a necessary condition for truthfulness of Myerson's mechanism without an additional ironing procedure.

2.4.3. Settings with affiliation. Settings which arise by imposing *affiliation* — a form of positive correlation — on the joint signal distribution F with density f. Affiliation was introduced by Milgrom and Weber [1982] and since then has become a standard assumption in the context of correlated and interdependent values, so much so that it is considered "almost synonymous with dependence in auctions." [de Castro 2010]. It is related to many well-studied mathematical concepts such as association, the FKG inequality and log-supermodularity [Alon and Spencer 2008].

Intuitively, signals are affiliated when observing a subset of high signals makes it more likely that the remaining signals are also high. Formally, for every pair of signal profiles $\vec{s}, \vec{t} \in [0, \omega_i]^n$ it must hold that

$$f(\vec{s} \vee \vec{t}) f(\vec{s} \wedge \vec{t}) \geq f(\vec{s}) f(\vec{t}), \tag{4}$$

where $(\vec{s} \vee \vec{t})$ is the component-wise maximum, and $(\vec{s} \wedge \vec{t})$ is the component-wise minimum. Note that the inequality in (4) holds with equality for independent signals. It also holds for the multivariate normal distribution in Example 2.3, which is affiliated since all pairwise covariances are nonnegative [de Castro and Paarsch 2010]. An example showing that affiliation is a stronger condition than positive correlation is the bivariate uniform distribution over support $\{(1,2), (2,1), (3,3), (4,4), (5,5), (6,6)\}$; the two variables are positively correlated but not affiliated.

2.4.4. Symmetric settings. Symmetry involves assumptions on both valuation functions and the signal distribution: (a) For every bidder i it is assumed that $v_i(\vec{s}) = v(s_i, s_{-i})$, where v is common to all bidders and symmetric in its last $n-1$ arguments; (b) The joint density f is assumed to be defined on support $[0, \omega]^n$ and symmetric in all its arguments. In a symmetric setting bidders thus have the same conditional densities, revenue curves and virtual value functions. Their values may be different however, since their own signal plays a distinct role in the valuation function. Notably, Milgrom and Weber [1982] study symmetric settings with affiliation; a concrete example of such a setting is weighted-sum values (Example 2.1) with the multivariate normal signals of Example 2.3.

2.4.5. Settings with a single crossing condition. Settings in which a *single crossing* condition is imposed on either values or virtual values. Let $x_1(\cdot), \ldots, x_n(\cdot)$ be some functions (such as values or virtual values) of the signals; a single crossing condition captures the idea that bidder i's signal has a greater influence on x_i than on any other bidder's function x_j.[4] Formally, for every i and $j \neq i$, and for every $\vec{s}$,

$$\frac{\partial x_i}{\partial s_i}(\vec{s}) > \frac{\partial x_j}{\partial s_i}(\vec{s}). \tag{5}$$

Weaker versions of single crossing may require a non-strict inequality, or that the inequality hold only for $i, j, \vec{s}$ such that $x_i(\vec{s}) = x_j(\vec{s}) = \max_k\{x_k(\vec{s})\}$. Stronger versions may require the left-hand side of Equation 5 to be non-negative and the right-hand side non-positive (see, e.g., Lemma 5.4), or even that for every $\vec{s}$ such that $s_i > s_j$, $x_i(\vec{s}) > x_j(\vec{s})$ (see, e.g., Lemma 5.5). The weighted-sum values in Example 2.1 are weakly single crossing.

3. MECHANISM AND SOLUTION CONCEPTS

By the revelation principle, we focus without loss of generality on *direct mechanisms*, in which bidders directly report their private signals. An exception is the English auction discussed below (Section 3.3). We restrict attention to IC mechanisms and so make no distinction between reported and actual signals.

A (randomized) mechanism M consists of an *allocation rule* $x_i(\cdot)$ and a *payment rule* $p_i(\cdot)$ for every bidder i, where $x_i(\vec{s})$ is the probability over the internal randomness of the mechanism that bidder i wins given the other signals $\vec{s}$, and $p_i(\vec{s}))$ is the expected payment of bidder i given s_{-i}. If M is deterministic, $x_i(\vec{s}) \in \{0, 1\}$ and $p_i(\vec{s})$ is the actual payment.

We assume risk-neutral bidders with quasi-linear utilities, i.e., given a mechanism and a signal profile $\vec{s}$, bidder i's effective utility is $x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})$.

3.1. Solution Concepts

Mechanism design aims to define the rules of a game played by the bidders, such that a solution of the game has desirable properties, in particular good objective function value subject to IC and IR. The solution concept — what constitutes a solution to the game — dictates the possibilities and impossibilities of mechanism design theory [Chung and Ely 2006]. We now describe three major solution concepts corresponding to three different equilibria types of the game.[5]

Definition 3.1 (*Ex post IC and ex post IR mechanism*). A mechanism is ex post IC and ex post IR if for every bidder i, true signal s_i, false report $\tilde{s}_i$, and signal profile s_{-i},

$$x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s}) \;\geq\; x_i(\tilde{s}_i, s_{-i})v_i(\vec{s}) - p_i(\tilde{s}_i, s_{-i}); \tag{6}$$

$$x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s}) \;\geq\; 0. \tag{7}$$

(Inequality 6 is the ex post IC condition and Inequality 7 is the ex post IR condition.)

[4]The term “single crossing” comes from the fact that keeping all other signals fixed and varying only s_i, x_i as a function of s_i is “steeper” than x_j, so the two cross at most once.

[5]All IC and IR conditions hold in expectation over the internal randomness of the mechanism.

In words: participating and truthtelling is an ex post equilibrium of the corresponding game, that is, it is a Nash equilibrium in the ex post stage of the game where private signals are common knowledge.

Definition 3.2 (*Dominant strategy IC mechanism*). A mechanism is dominant strategy IC if for every bidder i, true signal profile $\vec{s}$, and reported signal profile $\vec{r}$,

$$x_i(s_i, r_{-i})v_i(\vec{s}) - p_i(s_i, r_{-i}) \geq x_i(\vec{r})v_i(\vec{s}) - p_i(\vec{r}),$$

i.e., truthtelling is a dominant-strategy equilibrium of the corresponding game.

Definition 3.3 (*Bayesian IC and interim IR mechanism*). A mechanism is Bayesian IC and interim IR if for every bidder i, true signal s_i, and false report $\tilde{s}_i$,

$$\mathbb{E}_{s_{-i}}[x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})] \;\geq\; \mathbb{E}_{s_{-i}}[x_i(\tilde{s}_i, s_{-i})v_i(\vec{s}) - p_i(\tilde{s}_i, s_{-i})];$$
$$\mathbb{E}_{s_{-i}}[x_i(\vec{s})v_i(\vec{s}) - p_i(\vec{s})] \;\geq\; 0.$$

That is: participating and truthtelling is a Bayes-Nash equilibrium of the corresponding game in the interim stage, in which each individual knows his own signal but not the others.

3.2. Discussion of the Ex Post Solution Concept

The above definitions show that ex post is a weaker solution concept than dominant strategies (for which truthfulness holds for any reported signal profile), and a stronger one than Bayesian/interim (whose guarantees are in expectation over the true signal profile). We now briefly discuss our choice to focus on this intermediate solution concept. For additional discussion see [Segal 2003; Milgrom 2004; Bergemann and Morris 2005; Chung and Ely 2006; 2007].

3.2.1. Ex post vs. Bayesian. The solution concept most widely used in mechanism design theory is Bayes-Nash equilibrium [Chung and Ely 2006]; in practice, the common first-price and second-price auctions in interdependent settings are Bayesian and not ex post. On the flip side, the Crémer-McLean mechanism has been criticized as impractical for, among other issues, lack of the ex post IR property. This makes the outcome instable in the sense that bidders may regret their participation in hindsight, and attempt to exercise their de facto “veto” power of walking away from the auction – refusing to collect their winnings and to honor their payments (cf., [Compte and Jehiel 2009]). The lack of ex post IR also requires the Crémer-McLean mechanism to rely on bidders’ knowledge of the joint distribution, without which they cannot determine whether participating is rational.

Focusing on ex post mechanisms prevents these issues. Ex post IC and ex post IR are “no regret” properties — for any realization of the signals, bidders regret neither participating in the auction nor reporting their signals truthfully, even when all signals become publicly known. This makes the mechanism more robust (and thus closer to the computer science worst-case approach). To decide whether to participate and how to report, bidders do not have to know the signal distribution, only the signal support and the valuation functions. This is compatible with Wilson’s doctrine of detail-free mechanisms that are robust to detailed knowledge of the distribution [Wilson 1987]. Among other advantages, robustness saves transaction costs associated with learning about opponents’ distributions, and also benefits the seller, who may be wary of using a Bayesian mechanism if unsure how well bidders are informed.

We now mention two caveats to the ex post approach. First, in settings such as the mineral rights model (Example 2.2), one can argue that a bidder’s knowledge of his own valuation function $v(\vec{s}) = \mathbb{E}_{V\sim F_V}[V \mid \vec{s}]$ depends on his knowledge of the others’ distributions — this is necessary for him to derive v from the publicly known distribution F_V. In a model that crucially depends on bidders' knowledge of each other's distributions, and assuming the seller is aware that bidders are well-informed, there is less added robustness in an ex post solution over a Bayesian one. Note however that this issue does not arise in settings such as Example 2.1, and also that it arguably indicates the "type space" is simply not rich enough (cf., [Bergemann and Morris 2005]). A second and related caveat is that it is debatable whether ex post is *necessary* for robustness; this question and more generally the theoretical foundation of robustness in mechanism design is discussed in [Segal 2003; Bergemann and Morris 2005; Chung and Ely 2006; 2007] and references within.

3.2.2. Ex post vs. dominant strategies. For *private* values (whether correlated or independent), dominant strategy IC and ex post IC coincide. For non-private values, however, the concept of dominant strategy IC guarantees an even stronger no regret property than the concept of ex post IC, since it does not depend on the other bidders reporting truthfully. For example, in the weighted-sum values case (Example 2.1), if bidder j under-reports his signal s_j, and bidder i somehow knows j's true signal, in an ex post mechanism i may potentially benefit by over-reporting his signal s_i, so that his true value is reflected by the mechanism.

The following example demonstrates that the dominant strategy IC requirement may be too strong for a deterministic mechanism to extract non-trivial revenue. The example involves (by necessity) non-private values, and shows there are cases in which there's a sensible ex post mechanism with positive revenue, whereas the only dominant strategy IC mechanism is unreasonable and has zero revenue.

Example 3.4. Two bidders compete for a single item. Their values are $v_1 = s_1 s_2$ and $v_2 = 0$, where $s_1, s_2 \in \{0, 1\}$. If one of the reported signals is 0, by ex post IR the mechanism gets zero revenue. For reported signal profile $\vec{r} = (1, 1)$, to achieve non-zero revenue the mechanism extracts from bidder 1 a payment bounded away from 0. However, if the true signal profile is $\vec{s} = (1, 0)$ but bidder 2 reports $r_2 = 1$, then bidder 1 is better off reporting $r_1 = 0$ untruthfully, in contradiction to dominant strategy IC.

3.3. The English Auction

The *English auction* is an ascending price auction that operates in "value space": bidders act upon their postulated values rather then report their signals to the mechanism. Specifically, in order to determine when to (irrevocably) drop out of the ascending auction, bidders must constantly update their conjectured value based on their observations up to the current point in the auction. All other mechanisms we consider are direct revelation mechanisms that work in "signal space", and the English auction provides an indirect implementation for them [Lopomo 2000; Chung and Ely 2007].

The most relevant version of the English auction for our purpose is the so-called "Japanese" version (for details see, e.g., [Krishna 2010]). In this version, the auctioneer gradually raises the price of the item for sale; once a bidder finds the price too high, he indicates (for instance by lowering his hand) that he is no longer participating in the auction. The winner pays the price at which the second-to-last bidder dropped out. The crucial aspect of the English action is that it is *open* – the prices at which bidders drop out are observed by all. The English auction has a unique (cf., [Bikhchandani et al. 2002]) symmetric equilibrium studied by Milgrom and Weber [1982]; thanks to the openness of the auction, this equilibrium has the remarkable property of ensuring no ex post regret.

4. RELATED WORK

In this section we discuss related work on revenue guarantees of auctions in single-parameter settings. Section 5 of this paper can be seen as unifying and generalizing many previous results dispersed across the mechanism design literature; we now survey these results as well as present previous work on computational aspects. Note that this section is not a preliminary to Section 5, which is self-contained.

In describing previous results and in Section 5 itself we use the following terminology: We refer to results similar to Proposition 5.1 as *characterization results*; these state necessary and sufficient conditions on the allocation and/or payment rules such that the resulting mechanism is IC and IR with respect to the desired solution concept. By *virtual surplus results* we mean results similar to Proposition 5.2, showing that revenue equals virtual surplus in expectation for an appropriate definition of virtual values. *Optimal mechanism results* state conditions under which the optimal mechanism can be derived from the virtual surplus results.

4.1. Independent Values, Bayesian Solution

Myerson's 1981 paper lays the foundations of optimal mechanism design: Myerson considers Bayesian IC and interim IR mechanisms in the IPV model, and establishes characterization, virtual surplus and optimal mechanism results. The optimal mechanism turns out to be deterministic, dominant strategy IC and ex post IR, while achieving optimality among all randomized, Bayesian IC and interim IR counterparts. A regularity condition simplifies the Myerson mechanism but is not required.

Myerson's characterization and virtual surplus (but not optimal mechanism[6]) results apply to non-private (independent) values as well, for an appropriately modified definition of virtual values [Bulow and Klemperer 1996; Klemperer 1999]. Additional work on optimal auctions in settings with non-private values includes Branco [1996], Ülkü [2012].

4.2. Interdependent Values, Bayesian Solution

Myerson's theory does not directly apply when there's correlation among the bidders. The complicating issue is that in the presence of correlation, the allocation and payment rules for a bidder may depend not only on his reported signal, but also on his true signal through correlation with other bidders' signals. Cremér and McLean [1985; 1988] design an ex post IC but interim IR auction, which extracts full welfare in expectation under a mild "full rank" condition on the correlation; their mechanism is a generalized VCG auction, augmented with carefully designed lotteries. McAfee and Reny [1992] extend this result from discrete to continuous signals.

In a classic paper, building upon early work by Wilson [1969], Milgrom and Weber [1982] lay out a general model of interdependent values, and develop the *linkage principle* in place of the revenue equivalence principle. They apply the linkage principle to rank the common auction formats (first-price, second-price, English and Dutch auctions) according to their expected revenue, when signals are affiliated and bids form a symmetric Bayes-Nash equlibrium (actually an ex post equilibrium for the English auction).

4.3. Interdependent Values, Ex Post Solution

The ex post solution concept has generated much interest in the last decade; we now survey several papers most related to our work.

[6]Remarkably, Myerson's model does allow for a natural but restricted form of non-private values.

Correlated values. Section 5 is closely related to the work of Segal [2003] (see also, [Chung and Ely 2007]). Segal studies ex post IC and ex post IR mechanisms for selling multiple units of an item in the correlated values model. He gives a characterization and virtual surplus result based on conditional virtual values as defined in Equation 2. Segal also derives an optimal mechanism result under regularity and the assumption that conditional virtual values are single crossing; he notes that for affiliated signals the latter assumption holds. Our results in Section 5 can be seen as a generalization of Segal's results beyond multi-unit settings and beyond private values.

Interdependent values. A characterization result for ex post IC and ex post IR mechanisms in the interdependent values model is found in Chung and Ely [2006] (see also [Lopomo 2000; Vohra 2011]). Chung and Ely's result is via an interesting connection among the following characterizations: ex post IC and ex post IR mechanisms for interdependent values, Bayesian IC and interim IR mechanisms for independent values, ex post IC (equivalently, dominant strategy IC) and ex post IR mechanisms for private values, and IC and IR mechanisms for a single bidder (for which all solution concepts converge).

For a single item, Vohra [2011] states virtual surplus and optimal mechanism results, where the former is with respect to conditional virtual values as defined in Equation 1, and the latter is under the assumption that conditional virtual values are single-crossing. Vohra notes that for this result to be useful, one must identify restrictions on the distribution and valuation functions that would lead to single crossing. Such restrictions appear in Section 5. Recently and concurrently to our work, Csapo and Muller [2013] and Li [2013] also develop virtual surplus results for interdependent values. Csapo and Muller apply these in the context of supplying a single public good and assuming discrete signals. A more detailed description of the work of Li appears in Section 4.6 below.

Beyond single-parameter. Jehiel et al. [2006] show impossibility results for ex post implementation in multi-parameter settings. In particular, in a public decision setting with generic valuations, the only deterministic social choice functions that are ex post implementable are trivial (i.e., constant).

4.4. English auction with interdependent values

The importance of the English auction in the context of non-IPV settings has long been recognized. For non-private values, Bulow and Klemperer study symmetric bidders under a strong regularity condition, and show that the English auction's expected revenue (with or without reserve) equals the expected conditional virtual surplus as defined in Equation 1 above [Bulow and Klemperer 1996, Lemmas 1 and 2]. They establish that the English auction with optimally-chosen reserve is optimal among all Bayesian IC and interim IR mechanisms [Bulow and Klemperer 1996, Theorem 2]. McAfee and Reny [1992] show that while the English auction with reserve is optimal in a symmetric independent setting, minor perturbations of the distribution can introduce correlation and destroy optimality in comparison to other Bayesian mechanisms. They conjecture that the English auction's prevelance in practice has to do with the need to perform well in a variety of circumstances, and call for formalizing a notion of robustness.

The ex post solution concept adopted in this paper is precisely such a robustness notion. The following two results are close to our work, and we rederive them as corollaries in our framework (see Corollaries 5.12 and 5.13). For correlated values and under regularity and affiliation assumptions, Chung and Ely [2007] show that the English auction with optimally-chosen reserve is optimal among all ex post IC and ex post IR mechanisms. For interdependent values in the Milgrom and Weber setting, Lopomo

[2000] identifies conditions under which the English auction with optimally-chosen reserve is optimal among all mechanisms in a class he terms as "no-regret" (see Definition 5.8).

For completeness we describe in more detail the result of Lopomo. He studies mechanisms with a "no-regret" equilibrium in the sense that each bidder has no incentive to revise his decisions after observing his opponents behavior; ex post direct revelation mechanisms are a subclass in which no regret holds after observing all signals. Lopomo shows that payments in a no-regret equilibrium must be determined by the allocation rule and by the bidders' willingness to pay given all information revealed by the others' actions. He then shows that for a fixed allocation rule and assuming affiliation, the expected revenue is maximized by revealing all information to the winning bidder. The next step is to express the expected revenue as conditional virtual surplus, from which the optimal allocation rule can be derived under additional assumptions. Lopomo then shows that in the Milgrom-Weber model, the English auction with reserve implements the optimal allocation rule. Lopomo also demonstrates that the English auction is not optimal among the wider class of interim IR mechanisms with a "losers do not pay" restriction.

4.5. Computational Considerations, Randomized and Near-Optimal Mechanisms

Oracle vs. explicit model. An alternative approach to characterizing ex post IC and ex post IR in order to find the optimal mechanism is designing a computationally-tractable algorithm that computes or approximates such a mechanism. This requires addressing the question of how to represent the joint signal distribution. In the *oracle model*, the distribution is available to the mechanism/algorithm as a black-box, which can be queried with respect to conditional probabilities; upon receiving a signal profile as input, the mechanism/algorithm submits queries and returns an allocation rule and payments. In the *explicit model*, the distribution is explicitly provided as input, upon which the algorithm outputs the mechanism's allocation and payment rules.

Note that a hardness result in the explicit model implies hardness in the oracle model, whereas a positive algorithmic result in the oracle model implies such a result in the explicit model.

Computational hardness. Papadimitriou and Pierrakos [2011] prove computational hardness in the explicit model – even for correlated values, finding the optimal *deterministic*, ex post IC and ex post IR mechanism when there are at least three bidders is NP-hard. When there are exactly two bidders, the optimal deterministic mechanism can be computed in polynomial time and is optimal among all randomized mechanisms as well [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011].

Randomized mechanisms. Computational hardness does not extend to optimal randomized mechanisms, which can be computed in the explicit model in polynomial time for all single-parameter domains, as well as unit-demand and additive multi-parameter domains [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011]. With at least three bidders, randomized mechanisms can strictly outperform deterministic mechanisms in terms of expected revenue, albeit by a small constant factor (see next paragraph). This implies that additional assumptions are needed for a Myerson-like deterministic mechanism to be optimal (see Section 5).

Near-optimal mechanisms. In the oracle model with correlated values, Ronen [2001] designs the *lookahead auction* — a simple, deterministic, ex post IC and ex post IR mechanism, which guarantees a constant approximation to the optimal expected revenue. Dobzinski et al. [2011] build upon Ronen's work to design, for single-item settings, a deterministic mechanism that achieves a $5/3$-approximation, and a randomized one that achieves a $(3/2 + \epsilon)$-approximation (for improved bounds see [Chen et al. 2011]).

Informational hardness for the English auction. In the oracle model, Ronen and Saberi [2002] show that a deterministic English auction cannot achieve an approximation ratio better than $3/4$ with respect to the optimal expected revenue. Due to the oracle setup, this bound is explicit and does not rely on complexity assumptions.

4.6. Applications

Myerson's theory has multiple applications in the IPV model. Examples of applications studied in the algorithmic game theory community include simple near-optimal auctions [Neeman 2003; Hartline and Roughgarden 2009], prior-independent mechanisms [Segal 2003; Dhangwatnotai et al. 2010], and prior-free mechanisms [Goldberg et al. 2006].

In Section 6, we expand upon the theme of robustness by developing prior-independent mechanisms for interdependent values, using techniques from Dhangwatnotai et al. [2010]. Independently from and orthogonal to our work, Li [2013] shows a simple near-optimal auction for settings with interdependent values. She studies the VCG mechanism with monopoly reserves in matroid settings, where values satisfy a single crossing condition and the valuation distribution satisfies the generalized monotone hazard rate condition. Li shows that in expectation, VCG with monopoly reserves extracts at least $1/e$ of the full surplus.

5. MYERSON THEORY FOR INTERDEPENDENT VALUES

The fundamental results of single-parameter optimal auction theory — Myerson's optimal mechanism and characterization results leading to it — do not carry over to interdependent settings. In this section we show that these results are at least partially recovered with small adaptations once we impose the ex post requirements.

The intuition behind this finding is as follows. The original proofs rely on signal independence so that both the probability x_i of winning and the expected payment p_i depend only on bidder i's reported signal, not on his true one. By switching from Bayesian IC and interim IR to ex post IC and ex post IR, we ensure the guarantees hold for any signal profile s_{-i}. Since we can now fix s_{-i}, rules x_i and p_i once again depend only on bidder i's reported signal, and so the independence assumption is no longer necessary.

We describe the organization of our results using the terminology introduced in Section 4. Section 5.1 develops characterization and virtual surplus results, and Sections 5.2 and 5.3 establish optimal mechanism results (Section 5.2 addresses correlated values while Section 5.3 deals with full interdependence). For completeness, Section 5.4 discusses indirect implementation by the English auction. Section 5.5 discusses the assumptions of single-crossing and regularity that are used in the optimal mechanism results of 5.2 and 5.3. Many of the proofs are deferred to Appendix A.

5.1. Characterization and Expected Revenue of Ex Post Mechanisms

We begin by developing the theory as far as we can with no assumptions on the setting, i.e., we refrain from adding any of the constraints in Section 2.4. We show that characterization and equal-revenue results hold.

PROPOSITION 5.1 (CHARACTERIZATION). *For every interdependent values setting, a mechanism is ex post IC and ex post IR if and only if for every i, s_{-i}, the allocation rule x_i is monotone non-decreasing in the signal s_i, and the following payment identity*

ALGORITHM 1: Myerson Mechanism for Interdependent Values

(1) Elicit signal reports $\vec{s}$ from the bidders.
(2) Maximize the conditional virtual surplus by allocating to the feasible set S with the highest non-negative conditional virtual value $\sum_{i\in S}\varphi_i(s_i \mid s_{-i})$, breaking ties arbitrarily but consistently.
(3) Charge every winner i a payment $p_i(\vec{s}) = v_i(s_i^*, s_{-i})$, where s_i^* is the threshold signal such that given the other signals s_{-i}, if i's signal were below the threshold he would no longer win the auction.

and payment inequality hold:

$$\begin{aligned} p_i(\vec{s}) \;&=\; x_i(\vec{s})v_i(\vec{s}) - \int_{v_i(0,s_{-i})}^{v_i(s_i,s_{-i})} x_i\left(v_i^{-1}(t \mid s_{-i}), s_{-i}\right) dt \\ &\quad - \left(x_i(0,s_{-i})v_i(0,s_{-i}) - p_i(0,s_{-i})\right); \\ p_i(0,s_{-i}) \;&\le\; x_i(0,s_{-i})v_i(0,s_{-i}). \end{aligned}$$

PROOF. See Appendix A. □

The payment identity and inequality imply that the allocation rule for every bidder determines the bidder's payment up to his expected payoff for a zero signal, and that this expected payoff must be non-negative. For private values, with a standard assumption of no positive transfers, the payment constraints simplify to the identity

$$p_i(\vec{s}) = x_i(\vec{s})s_i - \int_0^{s_i} x_i\left(t, s_{-i}\right) dt.$$

PROPOSITION 5.2 (REVENUE EQUALS VIRTUAL SURPLUS IN EXPECTATION). *For every interdependent values setting, the expected revenue of an ex post IC and ex post IR mechanism equals its expected conditional virtual surplus, up to an additive factor:*

$$\begin{aligned} \mathbb{E}_{\vec{s}}\left[\sum_i p_i(\vec{s})\right] \;&=\; \mathbb{E}_{\vec{s}}\left[\sum_i x_i(\vec{s})\varphi_i(s_i \mid s_{-i})\right] \\ &\quad - \sum_i \mathbb{E}_{s_{-i}}\left[\left(x_i(0,s_{-i})v_i(0,s_{-i}) - p_i(0,s_{-i})\right)\right] \end{aligned}$$

PROOF. See Appendix A. □

Note that the additive term is just minus the sum of the bidders' expected payoffs for zero signals. For private values this term disappears.

5.2. Optimal Mechanism for Correlated Private Values

Proposition 5.2 suggests that to optimize expected revenue, the best course of action is to maximize conditional virtual surplus pointwise. However the issue is monotonicity – even in the independent private values model, regularity is necessary for pointwise maximization to form a monotone allocation rule, and in more general models we need more assumptions (see discussion in Section 5.5).

In this section we focus on correlated values in matroid settings, with assumptions of regularity and affiliation (recall Sections 2.4.1, 2.4.2 and 2.4.3). An example of such a setting – which is symmetric in addition to regular and affiliated – is a single-item setting where bidders' values are drawn from the multivariate normal distribution in Example 2.3.

In Algorithm 1 we define a Myerson mechanism for interdependent values. The main result in this section is its optimality for correlated values under the above assumptions.

THEOREM 5.3 (MYERSON MECHANISM IS EX POST IC, IR AND OPTIMAL). *For every matroid setting with correlated values that satisfies regularity and affiliation, the Myerson mechanism is ex post IC, ex post IR, and optimal among all ex post IC and ex post IR mechanisms.*

The following lemma is key to our analysis of the Myerson mechanism's performance.

LEMMA 5.4 (SINGLE CROSSING OF CONDITIONAL VIRTUAL VALUES). *For every correlated values setting with regular affiliated distribution, raising signal s_i weakly increases bidder i's conditional virtual value, and weakly decreases all other conditional virtual values.*

PROOF. First note that by regularity of the signal distribution, raising signal s_i increases bidder i's conditional virtual value $\varphi_i(s_i \mid s_{-i})$. It is left to prove the following claim: For every two bidders i, j, and every signal profile s_{-i}, bidder j's conditional virtual value is weakly decreasing in bidder $i's$ signal s_i.

Let $\tilde{s}_i \geq s_i$, and denote by $\tilde{s}_{-j}, s_{-j}$ the signal profiles excluding j with i's signal set to $\tilde{s}_i, s_i$, respectively. By definition, $\tilde{s}_{-j} \geq s_{-j}$. We show that $\varphi_j(s_j \mid \tilde{s}_{-j}) \leq \varphi_j(s_j \mid s_{-j})$, where

$$\varphi_i(s_j \mid \tilde{s}_{-j}) \;=\; s_j - \frac{1 - F_j(s_j \mid \tilde{s}_{-j})}{f_j(s_j \mid \tilde{s}_{-j})};$$
$$\varphi_i(s_j \mid s_{-j}) \;=\; s_j - \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})}.$$

By affiliation, $F_j(s_j \mid \tilde{s}_{-j})$ *dominates* $F_j(s_j \mid s_{-j})$ *in terms of hazard rate* [Krishna 2010, Appendix D], i.e.,

$$\frac{1 - F_j(s_j \mid \tilde{s}_{-j})}{f_j(s_j \mid \tilde{s}_{-j})} \;\geq\; \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})},$$

and this is sufficient to complete the proof. □

We remark that an even stronger version of single crossing holds if bidders are symmetric.

LEMMA 5.5 (ORDER OF VIRTUAL VALUES MATCHES ORDER OF SIGNALS). *For every correlated values setting with symmetric bidders and regular affiliated distribution, for every signal profile $\vec{s}$ such that $s_i \geq s_j$, the bidder with higher signal has higher conditional virtual value*

$$\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j}).$$

PROOF. Given a signal profile $\vec{s}$ where $s_i \geq s_j$, we show that $\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j})$. Recall

$$\varphi_i(s_i \mid s_{-i}) \;=\; s_i - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})};$$
$$\varphi_j(s_j \mid s_{-j}) \;=\; s_j - \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})}.$$

The following three inequalities follow from regularity, Lemma 5.4 (single crossing conditional virtual values), and symmetry of the bidders and distributions, respectively.

These properties allow us to first replace s_i by $s_j \leq s_i$ in bidder i's virtual value, then compare bidder i's virtual value given signal s_j versus signal s_i for bidder j (i.e., replace s_{-i} by s_{-j}), and finally replace F_i, f_i by F_j, f_j, completing the proof:

$$\begin{aligned} s_i - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} &\geq s_j - \frac{1 - F_i(s_j \mid s_{-i})}{f_i(s_j \mid s_{-i})} \\ &\geq s_j - \frac{1 - F_i(s_j \mid s_{-j})}{f_i(s_j \mid s_{-j})} \\ &= s_j - \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})}. \end{aligned}$$

□

Next we establish that the allocation rule of the Myerson mechanism in Algorithm 1 is monotone in matroid settings.

LEMMA 5.6 (MONOTONICITY). *For every matroid setting with correlated values that satisfies regularity and affiliation, maximizing conditional virtual surplus is monotone.*

PROOF. In a matroid setting, maximizing conditional virtual surplus can be implemented by a greedy algorithm, which considers bidders in non-increasing order of conditional virtual values, and adds them to the winning set if their conditional virtual value is nonnegative and feasibility is maintained. By Lemma 5.4, raising signal s_i can only improve bidder i's ranking in the order of consideration, thus monotonicity holds. □

The example below demonstrates that the condition of a matroid setting in Lemma 5.6 is necessary.

Example 5.7 (*Non-monotonicity beyond matroids*). Consider a correlated values setting with three bidders. The signals are drawn from the following regular affiliated distribution: Signal profiles $(1,1,1), (1,2,1), (2,1,1), (2,2,1)$ appear with probabilities $0.4, 0.1, 0.1, 0.4$. Assume that the feasible sets of the single-parameter auction environment are bidder sets $\{1,2\}$ and $\{3\}$.

Now consider signal profile (1,1,1) and assume that bidder 1 raises his reported signal from 1 to 2. The bidders' conditional virtual value profile changes from $(0.75, 0.75, 1)$ to $(2, -3, 1)$. With the original signal reports, the feasible bidder set maximizing nonnegative conditional virtual surplus was $\{1,2\}$, whereas after bidder 1 raises his report it becomes $\{3\}$, contradicting monotonicity.

The problem arises since by raising his own signal, bidder 1 decreased the conditional virtual value of bidder 2, and so 1 and 2 no longer form the feasible set with highest non-negative conditional virtual surplus.

We our now ready to prove the main result regarding optimality of the Myerson mechanism.

PROOF OF THEOREM 5.3. By the characterization of ex post mechanisms (Proposition 5.1) applied to private values, for every bidder i it is sufficient to show that the allocation rule x_i is monotone in the signal s_i, and that the payment identity $p_i(\vec{s}) = x_i(\vec{s})s_i - \int_0^{s_i} x_i\,(t, s_{-i})\,dt$ holds. Lemma 5.6 establishes monotonicity, and the payment identity holds by the following argument. The Myerson mechanism is deterministic and so either $x_i(\vec{s}) = 0$ or $x_i(\vec{s}) = 1$. In the former case, by monotonicity $x_i\,(t, s_{-i}) = 0$ for every $t \leq s_i$, so both sides of the identity are equal to zero. In the latter case, since s_i^* is bidder i's threshold signal, the right-hand side is

$$s_i - \int_0^{s_i} x_i\,(t, s_{-i})\,dt = s_i - (s_i - s_i^*) = s_i^*,$$

and for private values s_i^* is precisely the payment $p_i(\vec{s})$ charged by the Myerson mechanism.

It is left to show optimality. The expected revenue of an ex post IC and ex post IR mechanism is its expected virtual surplus (Proposition 5.2), and the Myerson mechanism maximizes virtual surplus for every signal profile. □

5.3. Optimal Mechanism for Interdependent Values

The results for correlated values generalize to interdependent values, however this setting is harder and requires further assumptions. Indeed, recall that the general conditional virtual value form in Equation 1 includes two extra dependencies on other bidders' signals relative to the form in Equation 2 which applies to correlated values. The extra assumptions are needed to establish monotonicity of the allocation rule in Algorithm 1 despite these dependencies.

We adopt the setting studied by Lopomo [2000] in the context of the English auction; namely, the assumptions we impose on our auction setting are that bidders are symmetric and have affiliated signals (Sections 2.4.3 and 2.4.4), and that the following conditions on the valuation function and distribution hold.

Definition 5.8 (*Lopomo assumptions*).

(1) MHR setting (Section 2.4.2);
(2) Bidders with higher signals have higher values, i.e., strong single crossing of values (Section 2.4.5);
(3) Bidders with higher signals have lower sensitivity of their value to their signal, i.e., the partial derivative of v_i with respect to s_i is weakly decreasing in s_i, and weakly increasing in s_j for every other j.

For every signal profile $\vec{s}$ such that $s_i \geq s_j$, the Lopomo assumptions imply:

$$\begin{aligned} v_i(\vec{s}) \;&\geq\; v_j(\vec{s}); \\ \frac{d}{ds_i} v_i(\vec{s}) \;&\leq\; \frac{d}{ds_j} v_j(\vec{s}). \end{aligned} \tag{8}$$

An example of a symmetric affiliated setting where the Lopomo assumptions hold is a single-item setting with weighted-sum values (Example 2.1), and signals drawn from the multivariate normal distribution of Example 2.3. Note that Equation (8) holds whenever values are multilinear.[7]

We can now state this section's main result - an analogue of Theorem 5.3 for interdependent values, showing that the Myerson mechanism defined in Algorithm 1 is optimal.

THEOREM 5.9 (MYERSON MECHANISM IS EX POST IC, IR AND OPTIMAL). *For every symmetric matroid setting with interdependent values that satisfies affiliation and the Lopomo assumptions, the Myerson mechanism is ex post IC, ex post IR, and optimal among all ex post IC and ex post IR mechanisms.*

[7]Recall that a function is multilinear if it is separately linear in each one of its variables – weighted sums and products are examples.

The proof of Theorem 5.9, like the proof of its analogue Theorem 5.3, boils down to showing monotonicity of the Myerson mechanism. We now turn to establishing monotonicity, using that the the order of conditional virtual values coincides with the order of signals. This strong form of single crossing for conditional virtual values (cf., Lemmas 5.4 and 5.5) is stated in the following lemma, which can be viewed as a generalization of the same result in the IPV model for a symmetric setting that satisfies regularity.

LEMMA 5.10 (ORDER OF VIRTUAL VALUES MATCHES ORDER OF SIGNALS). *For every symmetric setting with interdependent values that satisfies affiliation and the Lopomo assumptions, for every signal profile $\vec{s}$ such that $s_i \geq s_j$, the bidder with higher signal has higher conditional virtual value*

$$\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j}).$$

PROOF. Given a signal profile $\vec{s}$ where $s_i \geq s_j$, we show that $\varphi_i(s_i \mid s_{-i}) \geq \varphi_j(s_j \mid s_{-j})$. Recall

$$\begin{aligned}
\varphi_i(s_i \mid s_{-i}) &= v(s_i, s_{-i}) - \frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} \cdot \frac{d}{ds_i} v(s_i, s_{-i}); \\
\varphi_j(s_j \mid s_{-j}) &= v(s_j, s_{-j}) - \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})} \cdot \frac{d}{ds_j} v(s_j, s_{-j}).
\end{aligned}$$

We now compare the right-hand-side terms of the above equations.

By the second Lopomo assumption, bidders with higher signals have higher values and so $v(s_i, s_{-i}) \geq v(s_j, s_{-j})$. By the third Lopomo assumption, bidders with higher signals have a lower sensitivity of their value to their own signal and so $0 \leq \frac{d}{ds_i} v(s_i, s_{-i}) \leq \frac{d}{ds_j} v(s_j, s_{-j})$ (using the assumption that values are increasing in signals).

It is left to show that $0 \leq \frac{1-F_i(s_i|s_{-i})}{f_i(s_i|s_{-i})} \leq \frac{1-F_j(s_j|s_{-j})}{f_j(s_j|s_{-j})}$. The following three inequalities follow from the first Lopomo assumption of MHR, the affiliation assumption and resulting hazard rate dominance [Krishna 2010, Appendix D], and the bidders' symmetry and symmetry of distributions, respectively. These properties allow us to first replace s_i by $s_j \leq s_i$ in bidder i's inverse hazard rate, then compare the hazard rates given signal s_j versus signal s_i for bidder j (i.e., replace s_{-i} by s_{-j}), and finally replace F_i, f_i by F_j, f_j to complete the proof:

$$\begin{aligned}
\frac{1 - F_i(s_i \mid s_{-i})}{f_i(s_i \mid s_{-i})} &\leq \frac{1 - F_i(s_j \mid s_{-i})}{f_i(s_j \mid s_{-i})} \\
&\leq \frac{1 - F_i(s_j \mid s_{-j})}{f_i(s_j \mid s_{-j})} \\
&= \frac{1 - F_j(s_j \mid s_{-j})}{f_j(s_j \mid s_{-j})}.
\end{aligned}$$

□

We remark that in addition to the set of conditions in the statement of Lemma 5.10, there are also different (incomparable) conditions that suffice to prove a form of single crossing for conditional virtual values. One example of an alternative set of sufficient conditions excluding bidder symmetry is affiliated signals and the Lopomo assumptions, where we strengthen the first Lopomo assumption to the *generalized MHR* assumption of Li [2013].[8]

Lemma 5.10 implies that the Myerson mechanism is monotone.

LEMMA 5.11 (MONOTONICITY). *For every symmetric matroid setting with interdependent values that satisfies affiliation and the Lopomo assumptions, maximizing conditional virtual surplus is monotone.*

PROOF. By Lemma 5.10, raising signal s_i only improves bidder i's ranking in the greedy order of consideration by conditional virtual value. This is sufficient for monotonicity by a similar argument to that in the proof of Lemma 5.6. □

Monotonicity is sufficient to now prove optimality of the Myerson mechanism.

PROOF OF THEOREM 5.9. By the characterization of ex post mechanisms (Proposition 5.1) applied to interdependent values, for every bidder i it is sufficient to show that the allocation rule x_i is monotone in the signal s_i, and that the payment identity and payment inequality hold. Lemma 5.11 establishes monotonicity. The payment inequality $p_i(0, s_{-i}) \leq x_i(0, s_{-i})v_i(0, s_{-i})$ holds with equality since if $x_i(0, s_{-i}) = 0$ then $p_i(0, s_{-i}) = 0$, and if $x_i(0, s_{-i}) = 1$ then $p_i(0, s_{-i}) = v_i(s_i^*, s_{-i})$ where $s_i^* = 0$. As for the payment identity, by determinism and monotonicity of the Myerson mechanism and assuming $x_i(\vec{s}) = 1$,

$$
\begin{aligned}
x_i(\vec{s})v_i(\vec{s}) - \int_{v_i(0,s_{-i})}^{v_i(s_i,s_{-i})} x_i\left(v_i^{-1}(t \mid s_{-i}), s_{-i}\right) dt \;&=\; v_i(\vec{s}) - (v_i(s_i, s_{-i}) - v_i(s_i^*, s_{-i})) \\
&=\; v_i(s_i^*, s_{-i}) \\
&=\; p_i(\vec{s})
\end{aligned}
$$

It is left to show optimality. By Proposition 5.2, the expected revenue of an ex post IC and ex post IR mechanism is equal to its expected virtual surplus up to an additive term $\mathbb{E}_{s_{-i}}[\sum_i (x_i(0, s_{-i})v_i(0, s_{-i}) - p_i(0, s_{-i}))]$. The Myerson mechanism maximizes the virtual surplus for every signal profile, and sets the non-negative additive term to zero, thus achieving optimality. □

5.4. Indirect Implementation via English Auction

For completeness, we conclude this section with results by Lopomo [2000] and Chung and Ely [2007] on implementing the optimal mechanism via the English auction with a carefully chosen reserve, when there's a single item for sale and the bidders are symmetric. The relation between the previous subsections and these results is the same as the relation between Myerson's original mechanism and the following well-known result in the IPV model – the second-price auction with an optimal reserve maximizes the expected revenue from selling a single item when bidders are symmetric and regularity holds. By replacing the second-price auction with the English auction, and setting the reserve price after all bidders but one have dropped out and revealed their information, we get an indirect implementation of the optimal mechanism that works directly in value space.

COROLLARY 5.12 ([CHUNG AND ELY 2007]). *For every symmetric single-item setting with correlated values that satisfies regularity and affiliation, the English auction with optimal reserve price is optimal among all ex post IC and ex post IR mechanisms.*

[8]The generalized MHR condition defined and utilized in [Li 2013] states that the second term in Equation 1 is decreasing in s_i for all i and s_{-i}.

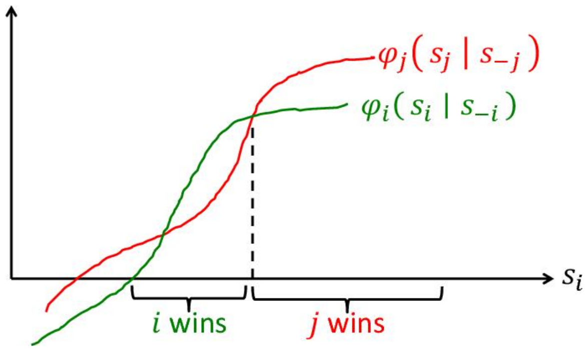

Fig. 1. Regularity without single crossing. The conditional virtual values of bidders i and j both increase as signal s_i increases, crossing each other more than once. Observe that if i and j are the two bidders with highest conditional virtual values, the virtual-surplus-maximizing allocation rule is not monotone.

PROOF. Consider the symmetric ex post equilibrium of the English auction [Milgrom and Weber 1982]. The last bidder to remain in the auction is the bidder with highest signal, who by Lemma 5.5 also has highest conditional virtual value. Set an optimal reserve price for this bidder. Since the reserve is set after signals of all other bidders are revealed, it guarantees that this bidder wins precisly when his conditional virtual value given the other signals is non-negative. The resulting mechanism thus maximizes conditional virtual surplus for every signal profile, and is equivalent to the optimal Myerson mechanism in Algorithm 1. □

The same proof with Lemma 5.5 replaced by Lemma 5.10 shows the following.

COROLLARY 5.13 ([LOPOMO 2000]). *For every symmetric single-item setting with interdependent values that satisfies affiliation and the Lopomo assumptions, the English auction with optimal reserve price is optimal among all ex post IC and ex post IR mechanisms.*

5.5. Discussion of Assumptions

The Myerson mechanism in Algorithm 1 is truthful for interdependent values only if its virtual-surplus-maximizing allocation rule is monotone. Even in the IPV model, a regularity assumption is necessary for the Myerson mechanism to be monotone without an additional *ironing* procedure (we dicuss ironing for interdependent values below). In the interdependent values model, an additional single crossing assumption for conditional virtual values is required. Figure 1 and Example 5.14 show what could go wrong if this assumption is violated.

Example 5.14 (*Regularity without single crossing in a wallet game*). Consider a symmetric, single-item setting with non-private values. Let the values be weighted-sum values with parameter $\beta = 1$ (Example 2.1), and let signals be independently drawn from the regular distribution $G(s) = 1 - s^{-2}$. Plugging into Equation 3, the conditional virtual value of bidder i is $\sum_j s_j - (1 - G(s_i))/g(s_i) = s_i/2 + \sum_{j \neq i} s_j$. Thus when bidder i's signal increases by Δs_i, his own conditional virtual value increases by $\Delta s_i/2$ while his competitors' conditional virtual values increase by the full difference Δs_i. Single crossing is thus violated and the Myerson mechanism will not be truthful.[9]

[9]In fact, in this example the order of conditional virtual values is exactly the order of signals *reversed*; we can show that the optimal allocation rule in this case is to pick a random winner – any other "more sensible" allocation rule will violate monotonicity.

Single crossing as an assumption in itself is quite opaque; above we have identified economically-meaningful conditions on the auction environment that are sufficient for single crossing to hold, both in the special case of correlated values and in the more general case of interdependent values. While the standard assumption of affiliation is sufficient in the correlated values setting, this is no longer the case for full interdependence. However, as we have seen, symmetry together with the Lopomo assumptions are sufficient (and alternative sufficient conditions exist as well). While the known computational hardness results imply that some of these assumptions (or alternative ones) are required for optimality of the deterministic Myerson mechanism, achieving a precise understanding of what is necessary remains an open question.

Ironing. Does the method of ironing developed by Myerson [1981] work for interdependent values?[10] Technically, the ironing method can easily be applied to conditional virtual values, i.e., the Myerson mechanism with ironing is well-defined for interdependent values. Furthermore, it still holds that ironed conditional virtual surplus gives an upper bound on conditional virtual surplus, which is tight for mechanisms that "respect" the ironed intervals (in the sense that the allocation does not change along such an interval).

The crucial difference from the IPV model is that the expected revenue of the Myerson mechanism with ironing can be *strictly lower* than the expected ironed conditional virtual surplus. Thus, even though the Myerson mechanism with ironing truthfully maximizes the latter, it is no longer guaranteed to achieve the maximum expected revenue. This gap arises due to the fact that the Myerson mechanism with ironing does not respect ironed intervals. Indeed, while the increase in a bidder's signal does not change his ironed conditional virtual value within an ironed segment, it may change others' ironed conditional virtual values, thus modifying the allocation.

Since the Myerson mechanism with ironing is deterministic, it is not surprising that ironing does not allow us to dispose altogether of assumptions on the valuations and/or distributions, as is the case in Myerson's paper (recall the negative results in [Dobzinski et al. 2011; Papadimitriou and Pierrakos 2011]). It is open whether ironing can help weaken these assumptions.

6. PRIOR-INDEPENDENCE FOR NON-PRIVATE VALUES

In this section we begin to develop a theory of prior-independence for interdependent values. Our main result is, for non-private values and the setting studied above (Section 5.3), a prior-independent mechanism that achieves a constant-factor approximation to the optimal expected revenue. An interesting direction for future work is to design good prior-independent mechanisms for general interdependent values.

This section is organized as follows: After presenting the setting and stating the main result, we prove our result for a simple single item setting in which bidders share a pure common value for the item. Section 6.5 generalizes the proof to matroid settings with non-private values.

6.1. Setting

We study a matroid setting with non-private values where signals are independent, and the Lopomo assumptions (Definition 5.8) hold.[11]

[10] A description of the ironing method is beyond the scope of this paper; for an introduction see, e.g., [Hartline 2012].

[11] Note that for digital goods settings, our results hold more generally; i.e., we no longer need all the Lopomo assumptions.

ALGORITHM 2: The Single Sample Mechanism for Interdependent Values

(1) Elicit signal reports $\vec{s}$ from the bidders.
(2) Choose a *reserve bidder* uniformly at random, denote his signal by s_r.
(3) Place the feasible set of non-reserve bidders with highest signals in a “potential winners” set P. Break ties arbitrarily but consistently.
(4) Allocate to every bidder $i \in P$ such that $s_i \geq s_r$.
(5) Charge every winner i a payment $v_i(\max\{s_r, t_i\}, s_{-i})$, where t_i is the threshold signal below which, given the signals of the other non-reserve bidders, i would not belong to P.

Symmetry. As is standard in the prior-independence literature (see, e.g., [Bulow and Klemperer 1996; Goldberg et al. 2006; Segal 2003; Dhangwatnotai et al. 2010]), we focus on symmetric environments with $n \geq 2$ bidders.

Notation. As above, let F denote the joint distribution of the independent signals. Let G be the distribution from which each of the i.i.d. signals is drawn, and let g be the corresponding density (G is the marginal distribution of the signals given the joint product distribution F). We denote by $G_{|s_{-i}}(\cdot), g_{|s_{-i}}(\cdot)$ the distribution and density of bidder i's value given the signal profile s_{-i} of the other bidders.

Remark 6.1 (*Strong MHR guarantee*). Observe that by independence, $G_{|s_{-i}}(\cdot)$ is simply the distribution of s_i. It follows that the inverse hazard rate of bidder i's value given signal profile $\vec{s}$ is

$$\frac{1 - G_{|s_{-i}}(v_i(\vec{s}))}{g_{|s_{-i}}(v_i(\vec{s}))} = \frac{1 - G(s_i)}{g(s_i)} \cdot \frac{d}{ds_i} v_i(\vec{s}). \tag{9}$$

By the first and third Lopomo assumptions, the inverse hazard rate in Equation 9 is weakly decreasing in s_i. We conclude that not only the signal distribution G is MHR, but also, for every bidder i, so is the value distribution $G_{|s_{-i}}$ for every signal profile s_{-i}.

6.2. The Single Sample Mechanism for Interdependent Values

We describe our prior-independent mechanism for interdependent values in Algorithm 2. It is a natural generalization of the *single sample* mechanism of Dhangwatnotai et al. [2010]. Observe that the mechanism makes no reference to the distribution G.[12]

We are now ready to state this section's main result – that the above prior-independent mechanism is near-optimal. We compare its expected revenue to OPT, the optimal expected revenue achieved by the generalization of Myerson's mechanism to interdependent values (Algorithm 1). In fact, due to the MHR setting, the proof will be able to relate the expected revenue to the expected *welfare*, establishing a stonger property of *effectiveness* as defined by Neeman [2003].

THEOREM 6.2 (SINGLE SAMPLE IS NEAR-OPTIMAL). *Let $n \geq 2$ and consider a symmetric matroid setting with non-private values in which the Lopomo assumptions hold. The prior-independent single sample mechanism in Algorithm 2 yields a constant factor approximation to* OPT.

[12] We remark that the mechanism is assumed to know the valuation function v_i. An intriguing open problem is to design a mechanism, perhaps based on an ascending or multi-stage auction, for which this assumption can be dropped.

6.3. Useful Properties of MHR Distributions

We motivate our focus on MHR settings by the following example, which shows that unlike independent private values for which regularity suffices, for interdependent values a stronger MHR assumption is necessary to guarantee near-optimality of the single sample mechanism. Note that unlike previous sections, the MHR assumption is for the approximation guarantee (weaker assumptions are sufficient for incentives only). More generally, the example demonstrates how interdependence can pose new technical challenges, arising from the information externalities among bidders. The original analysis of Dhangwatnotai et al. [2010] no longer applies; we present properties of MHR distributions that will be useful in the new analysis.

Example 6.3 (*Non-MHR setting*). Consider a digital goods setting with two bidders, whose i.i.d. signals are drawn from the equal revenue distribution $G(s) = 1 - (1/s)$, truncated to a finite range $[1, H]$, where H is an arbitrarily large constant. The bidders have weighted-sum values with $\beta = 1$, i.e., their pure common value is $s_1 + s_2$ (see Example 2.1). The optimal expected revenue in this setting is at least $\mathbb{E}[s_1] + \mathbb{E}[s_2] \approx 2\ln H$, by charging each bidder the signal of the other. However, the expected revenue of the single sample mechanism is $2\mathbb{E}[\min\{s_1, s_2\}]$, which is $\ll H$.

The gap in Example 6.3 between the expectation of the distribution and the expectation of the lower among two random samples is due to the long tail of the non-MHR equal revenue distribution; we now show that for MHR distributions this issue does not arise. The next lemma is tight for the exponential distribution.

LEMMA 6.4 (LOWER AMONG TWO SAMPLES FROM MHR DISTRIBUTION). *Let s, s' be two i.i.d. samples drawn from an MHR distribution, then*

$$\mathbb{E}[s \mid s \leq s'] \geq \frac{1}{2}\mathbb{E}[s].$$

PROOF. Let G with density g be the MHR distribution. Let $h(\cdot)$ be the hazard rate function of G and let $H(\cdot)$ be its cumulative hazard rate, i.e., $h(s) = g(s)/(1 - G(s))$, and $H(s) = \int_0^s h(z)dz$. By definition of the hazard rate function, $1 - G(s) = \exp(-H(s))$. Since G is MHR, $h(\cdot)$ is non-negative and weakly increasing, therefore $H(\cdot)$ is weakly increasing and convex. We can now write

$$\mathbb{E}[s \mid s \leq s'] = \int_0^\infty e^{-2H(s)} \geq \int_0^\infty e^{-H(2s)} \geq \frac{1}{2}\int_0^\infty e^{-H(s)} = \frac{1}{2}E[s],$$

where the first equality is by plugging in the distribution of the lower among two samples into $\mathbb{E}[x] = \int_0^\infty 1 - F(z)dz$, the first inequality is by convexity of H, and the second inequality is via integration by substitution. □

We now state a version of the previous lemma with an added threshold (and a necessarily slightly increased constant).

LEMMA 6.5 (LOWER AMONG TWO SAMPLES WITH THRESHOLD). *Let s, s' be two i.i.d. samples drawn from an MHR distribution. For every threshold $t \geq 0$,*

$$\mathbb{E}[\max\{s, t\} \mid \max\{s, t\} \leq s'] \geq \frac{1}{3}\mathbb{E}[s \mid s \geq t].$$

PROOF. Recall we'd like to show that

$$\mathbb{E}[\max\{s, t\} \mid \max\{s, t\} \leq s'] \geq \frac{1}{3}\mathbb{E}[s \mid s \geq t],$$

where s, s' are i.i.d. samples from an MHR distribution and t is a threshold. Since s is drawn from an MHR distribution we can write

$$\frac{1}{3}(t + \mathbb{E}[s]) \geq \frac{1}{3}\mathbb{E}[s \mid s \geq t].$$

Assume first that $t \leq \mathbb{E}[s]/2$, then

$$\mathbb{E}[\max\{s,t\} \mid \max\{s,t\} \leq s'] \geq \mathbb{E}[s \mid s \leq s'] \geq \mathbb{E}[s]/2 \geq \frac{1}{3}(t + \mathbb{E}[s]),$$

where the second inequality is by Lemma 6.4 and the last one is by assumption. We now turn to the case in which $t > \mathbb{E}[s]/2$, and complete the proof by observing that

$$\mathbb{E}[\max\{s,t\} \mid \max\{s,t\} \leq s'] \geq t \geq \frac{1}{3}(t + \mathbb{E}[s]).$$

□

The following simple lemmas are stated without proofs.

LEMMA 6.6 (CONDITIONAL MHR DISTRIBUTION). *Let signal s be randomly drawn from an MHR distribution. For any threshold t, the distribution of s conditional on $s \geq t$ is also MHR.*

LEMMA 6.7 (OPTIMAL REVENUE APPROXIMATES WELFARE [HARTLINE 2012]). *In a single bidder setting where the bidder's value is drawn from an MHR distribution, the optimal expected revenue is at least a $1/e$ fraction of the expected value.*

LEMMA 6.8 (MEDIAN AS RESERVE FOR REGULAR DISTRIBUTIONS [AZAR ET AL. 2013]). *In a single bidder setting where the bidder's value is drawn from a regular distribution G, a reserve price equal to the median of G guarantees at least half the optimal expected revenue, and the optimal expected revenue is thus at most the median.*

6.4. Proof for Simple Setting: Single Item with Common Value

In this section we prove Theorem 6.2 for a simple setting with a single item for sale, which is valued the same by all bidders. We first state and prove our main lemma.

Main lemma. Let s_1, s_2 be i.i.d. signals drawn from an MHR distribution G. Consider a single bidder with value $v(s_1, s_2)$, where v is a symmetric valuation function increasing in its arguments. Fixing signal s_1 (resp., s_2), let the value distribution $G_{|s_1}$ (resp., $G_{|s_2}$) be an MHR distribution. Let $c = 1/8e$ (where e is the base of the natural logarithm).[13]

LEMMA 6.9 (REUSING SAMPLE APPROXIMATES WELFARE).

$$\mathbb{E}_{s_1,s_2}[v(s_2, s_2) \mid s_2 \leq s_1] \geq c\mathbb{E}_{s_1,s_2}[v(s_1, s_2)]. \tag{10}$$

In words, plugging in the lower among s_1, s_2 into the valuation function decreases the expected value by a factor of no more than c.

PROOF. Let m be the median of distribution G. We begin with the left-hand side of Equation 10 and condition on the event that $s_2 \geq m$, which given that $s_2 \leq s_1$ occurs with probability $1/4$. Using that v is non-decreasing,

$$\begin{aligned}\mathbb{E}_{s_1,s_2}[v(s_2, s_2) | s_2 \leq s_1] &\geq \Pr_{s_1,s_2}[s_2 \geq m \mid s_2 \leq s_1]\mathbb{E}_{s_1,s_2}[v(s_2, s_2) \mid m \leq s_2 \leq s_1] \\ &\geq \frac{1}{4}\mathbb{E}_{s_1,s_2}[v(m, s_2) \mid s_2 \leq s_1].\end{aligned}$$

[13] We do not optimize the constant c.

Now we know that replacing a random MHR sample with the lower among two samples results in a loss of at most $1/2$ (Lemma 6.4). This can be applied to the distribution $G_{|s_1=m}$ of $v(m,\cdot)$, which is MHR by assumption. We get that

$$\mathbb{E}_{s_1,s_2}[v(m,s_2) \mid s_2 \le s_1] \ge \frac{1}{2}\mathbb{E}_{s_2}[v(m,s_2)].$$

Now fix s_2. Observe that by the signal independence assumption, $v(m,s_2)$ is the median of the MHR distribution $G_{|s_2}$ of $v(\cdot,s_2)$. Thus

$$v(m,s_2) \ge \frac{1}{e}\mathbb{E}_{s_1}[v(s_1,s_2)],$$

where the inequality follows by combining Lemma 6.8, by which the median upper bounds the optimal expected revenue, with Lemma 6.7, by which the optimal expected revenue and expected welfare are close. Taking expectation over s_2 completes the proof. □

We are now ready to prove the special case of our main theorem – near-optimality of the single sample mechanism for the simple single-item, common value setting.

PROOF OF THEOREM 6.2 FOR SINGLE ITEM WITH COMMON VALUE. Let $\tilde{v}(\vec{s})$ be the pure common value of the item for the bidders, whose i.i.d. signals $s_1,\ldots,s_n$ are drawn from an MHR distribution $\tilde{G}$. In this simple setting, the single sample mechanism in Algorithm 2 reduces to the following mechanism: choose a random reserve bidder; with probability $(n-1)/n$ the bidder with highest signal is not chosen as reserve; he then wins the item and is charged *according to the second highest signal*. We claim that in expectation, the revenue achieved by this VCG-like mechanism is a $c(n-1)/n$-fraction of the expected welfare $\mathbb{E}_{\vec{s}}[\tilde{v}(s_1,\ldots,s_n)]$, where c is as above.

The proof is by reduction to the single bidder setting of Lemma 6.9 (where the single bidder will correspond to the highest bidder). From now on we condition on the highest bidder not being chosen as reserve, incurring a loss of $(n-1)/n$.

Fix the $n-2$ lowest signals, denoted without loss of generality $s_3 \ge \cdots \ge s_n$. Let $v(\cdot,\cdot) = \tilde{v}(\cdot,\cdot,s_3,\ldots,s_n)$ be the valuation function given the fixed signals. Let G be the distribution $\tilde{G}$ conditioned on exceeding the threshold s_3. By Lemma 6.6, G is MHR, and the two highest signals s_1,s_2 can be seen as i.i.d. random draws from G. One of these is the second highest signal, and so we can write the expected revenue of the single sample mechanism as

$$\mathbb{E}_{s_1,s_2\sim G}[v(s_2,s_2) \mid s_2 \le s_1]. \tag{11}$$

In order to apply the main lemma (Lemma 6.9) to lower bound the expected revenue in 11, recall that $\tilde{v}$ and hence v are symmetric and increasing. It is left to show that $G_{|s_1}$ and $G_{|s_2}$ are MHR. Without loss of generality consider $G_{|s_2}$. We know from Remark 6.1 that given $s_2,\ldots,s_n$, the distribution of $\tilde{v}_{|s_{-1}}$ is MHR. If we condition this distribution on $\tilde{v}_{|s_{-1}}$ being at least as high as $\tilde{v}_{|s_{-1}}(s_3)$, we still get an MHR distribution by Lemma 6.6. The resulting distribution is precisely the distribution $G_{|s_2}$.

The proof can now be completed by applying Lemma 6.9 to get that the expected revenue in (11) is at least $c\mathbb{E}_{s_1,s_2\sim G}[v(s_1,s_2)]$ for any fixed profile $s_3,\ldots,s_n$, and finally by taking expectation over $s_3,\ldots,s_n$ according to the joint distribution of the $n-2$ lowest among n draws from $\tilde{G}$. □

6.5. Proof for General Setting

In this section we prove Theorem 6.2 for a general matroid setting in which bidders have symmetric but distinct values. The proof relies on an extension of the main lemma (Lemma 6.9) above.

As above, let s_1, s_2 be i.i.d. signals drawn from an MHR distribution G. Consider a single bidder with a symmetric and increasing valuation $v(s_1, s_2)$, and assume the value distributions $G_{|s_1}$ and $G_{|s_2}$ when one of the signals is fixed are MHR. Let $c' = 1/12e$.[14]

LEMMA 6.10 (REUSING SAMPLE WITH THRESHOLD). *For every threshold $t \geq 0$,*

$$\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},s_2) \mid \max\{s_2,t\} \leq s_1] \geq c'\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid t \leq s_1].$$

PROOF. Recall we want to show that, for $c' = 1/12e$,

$$\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},s_2) \mid \max\{s_2,t\} \leq s_1] \geq c'\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid t \leq s_1].$$

Let m be the median of distribution G from which s_1, s_2 are independently drawn. Similarly to the proof of Lemma 6.9 we have that

$$\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},s_2) \mid \max\{s_2,t\} \leq s_1] \geq \frac{1}{4}\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},m) \mid \max\{s_2,t\} \leq s_1].$$

By Lemma 6.5 applied to the distribution $G_{|s_2=m}$ of $v(\cdot, m)$, which is MHR by assumption, and using that v is weakly increasing, we get

$$\begin{aligned} \frac{1}{4}\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},m) \mid \max\{s_2,t\} \leq s_1] &= \\ \frac{1}{4}\mathbb{E}_{s_1,s_2}[\max\{v(s_2,m),v(t,m)\} \mid \max\{v(s_2,m),v(t,m)\} \leq v(s_1,m)] &\geq \\ \frac{1}{12}\mathbb{E}_{s_1}[v(s_1,m) \mid s_1 \geq t]. \end{aligned}$$

Similarly to the proof of Lemma 6.9, we now fix $s_1 \geq t$. Observe that $v(s_1, m)$ is the median of the MHR distribution $G_{|s_1}$ of $v(s_1, \cdot)$. Thus

$$v(s_1,m) \geq \frac{1}{e}\mathbb{E}_{s_2}[v(s_1,s_2)],$$

and taking expectation over s_1 conditional on $s_1 \geq t$ completes the proof. □

The proof of Theorem 6.2 for the general setting is by reduction to the single bidder setting and application of Lemma 6.10.

PROOF OF THEOREM 6.2 FOR GENERAL SETTING. Recall we wish to analyze the expected revenue of the single sample mechanism in Algorithm 2, for a symmetric matroid setting with $n \geq 2$ bidders, where the i.i.d. signals $s_1, \ldots, s_n$ are drawn from an MHR distribution G. Denote by $\tilde{v}$ the symmetric valuation function of the bidders. Similarly to the proof in Section 6.4, we will reduce this setting to a single bidder setting to which Lemma 6.10 is applicable.

Without loss of generality we name the chosen reserve bidder "bidder 2", and consider an arbitrary non-reserve bidder "bidder 1". We condition on the signals of all bidders other than 1 and 2 and omit them from the notation, i.e., we use the notation $v(s_1, s_2)$ for bidder 1's value. In addition, we denote by t the threshold for bidder 1 to

[14] As before, the constant is not optimized.

belong in the potential winner set P given the fixed signals. We can now write the expected contribution of bidder 1 to the expected revenue of the single sample mechanism as

$$\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},s_2) \mid \max\{s_2,t\} \leq s_1] \Pr[\max\{s_2,t\} \leq s_1]. \tag{12}$$

In what follows we show that the expected contribution in 12 is a constant fraction of the expected contribution of bidder 1 to the expected maximum welfare excluding bidder 2. To see how this completes the proof, take expectation over the fixed signals and sum up over non-reserve bidders. The total expected revenue of the single sample mechanism is thus a constant fraction of the welfare excluding the reserve bidder, which is in turn a $(n-1)/n$ fraction of the total welfare.

We begin by writing down the expected contribution of bidder 1 to the expected maximum welfare excluding bidder 2. Crucially, the same threshold t as in the single sample mechanism is the threshold for bidder 1 to be included in the welfare-maximizing set of bidders. This is because the potential winner set P of the single sample mechanism is the feasible set of non-reserve bidders with highest signals, and by single-crossing of values (second Lopomo assumption) and the matroid setting, this is also the welfare-maximizing feasible set. The expected welfare contribution is thus

$$\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid s_1 \geq t] \Pr[s_1 \geq t]. \tag{13}$$

It remains to compare (12) to (13). Since we know that v is symmetric and increasing and that $v(s_1,\cdot), v(\cdot,s_2)$ are distributed according to an MHR distribution (Remark 6.1), we can apply Lemma 6.10 to get

$$\begin{aligned}
\mathbb{E}_{s_1,s_2}[v(\max\{s_2,t\},s_2) \mid \max\{s_2,t\} \leq s_1] \Pr[\max\{s_2,t\} \leq s_1] &\geq \\
c'\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid s_1 \geq t] \Pr[s_1 \geq t] \Pr[s_1 \geq s_2 \mid s_1 \geq t] &\geq \\
c'\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid s_1 \geq t] \Pr[s_1 \geq t] \Pr[s_1 \geq s_2] &= \\
(c'/2)\mathbb{E}_{s_1,s_2}[v(s_1,s_2) \mid s_1 \geq t] \Pr[s_1 \geq t].&
\end{aligned}$$

□

7. OPEN QUESTIONS

Interdependent values are potentially a new frontier for algorithmic mechanism design; we conclude with a non-exhaustive list of directions for further research.

(1) Ex post optimal and near-optimal mechanism design:
   (a) Optimality: For correlated values, in the absence of regularity and single crossing, finding the optimal mechanism may be a computationally hard problem. Are there weaker conditions (e.g., just single crossing but not regularity) under which a meaningful description of the optimal mechanism is available, perhaps via some form of Myerson-inspired ironing (cf., [Myerson 1981])? Is randomness necessary to achieve optimality (cf., irregular non-single crossing example requiring randomness in [Dobzinski et al. 2011])? Similar questions apply to interdependent values, where the goal would also be to weaken the Lopomo assumptions. Identifying tractable settings for the optimal mechanism problem may also have direct applicability to the efficient mechanism problem.
   (b) Approximation: For correlated values, Dobzinski et al. [2011] show a near-optimal mechanism that does not rely on regularity or affiliation assumptions. Their mechanism is based on the appealing *lookahead* mechanism of Ronen [2001]. Is there a parallel result for interdependent values?
   (c) Simple, natural and practical mechanisms (as advocated in [Hartline and Roughgarden 2009]). For example, when the English auction is not optimal,

under what conditions is it approximately-optimal? See Li [2013] for one set of sufficient conditions.

(2) Beyond ex post mechanisms: There is a range of robustness levels to explore in mechanism design, and it is far from clear what is the "right" level. This may depend on information asymmetries in the market (e.g., is it reasonable/necessary to assume public knowledge of valuation functions), as well as on models of players' risk-averseness and on practical considerations, such as the difficulty to collect payments from losing bidders (e.g., this motivates the "losers do not pay" condition considered by Lopomo [2000]). What are the most economically-meaningful robustness requirements, and what are optimal and approximately-optimal mechanisms that achieve them?

(3) Beyond optimal mechanism design: Interdependent values are an active research area in the design of other economic mechanisms and markets, to which computational insight potentially has much to contribute – see for example Satterthwaite et al. [2011] on double auctions, Che and Kim [2012] on house allocation or Csapó and Müller [2013] on public goods.

ACKNOWLEDGMENTS

The authors wish to thank Moshe Babaioff, Paul Milgrom, Uri Nadav and the anonymous reviewers for helpful comments, as well as participants of the Market Design course at Stanford, the EconCS seminar at Harvard and the Theory seminar at Northwestern. In an earlier version of this work we presented Corollary 5.12 unaware that it appeared in a paper of Chung and Ely [2007]; we are grateful to Gabriel Carroll for pointing us to their paper.

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A. MISSING PROOFS RELATED TO MYERSON THEORY (SECTION 5)

PROOF OF PROPOSITION 5.1 (FOLLOWING [NISAN 2007], THEOREM 9.39). We fix i, s_{-i} and omit them from the notation for simplicity.

First direction. Assume x is monotone and the payment identity and inequality hold. For ex post IC we need to show that $x(s)v(s) - p(s) \geq x(s')v(s) - p(s')$ for every $s \neq s'$.

Plugging in the payment identity, we need to show

$$\int_{v(0)}^{v(s)} x\left(v^{-1}(t)\right) dt \geq x(s')\left(v(s) - v(s')\right) + \int_{v(0)}^{v(s')} x\left(v^{-1}(t)\right) dt.$$

If $s' > s$, this is equivalent to $x(s')\left(v(s') - v(s)\right) \geq \int_{v(s)}^{v(s')} x\left(v^{-1}(t)\right) dt$, which holds by monotonicity of x, and if $s > s'$ a similar argument applies. For ex post IR we need to show $\int_{v(0)}^{v(s)} x\left(v^{-1}(t)\right) dt + x(0)v(0) - p(0) \geq 0$, and this holds since x is non-negative and $x(0)v(0) \geq p(0)$.

Second direction. Assume ex post IR. So for every $s' > s$, $x(s)v(s) - p(s) \geq x(s')v(s) - p(s')$ and $x(s)v(s') - p(s) \leq x(s')v(s') - p(s')$. Subtracting the inequalities we get $x(s)\left(v(s) - v(s')\right) \geq x(s')\left(v(s) - v(s')\right)$, which implies monotonicity by the assumption that $v(\cdot)$ is strictly increasing in its argument. To derive the payment identity, we re-arrange the inequalities to get

$$\left(x(s') - x(s)\right) v(s) \leq p(s') - p(s) \leq \left(x(s') - x(s)\right) v(s').$$

We now set $s' = s + \epsilon$, divide throughout by ϵ and take the limit. We get $\frac{dp}{ds} = v(s)\frac{dx}{ds}$, and so $p(s) = p(0) + \int_0^s x'(t)v(t)dt$. Integration by parts gives $p(s) = p(0) + x(s)v(s) - x(0)v(0) - \int_0^s x(t)v'(t)dt$, and substituting $u = v(t)$ yields $p(s) = p(0) + x(s)v(s) - x(0)v(0) - \int_{v(0)}^{v(s)} x\left(v^{-1}(u)\right) du$. To derive the payment inequality we apply the ex post IR assumption for $s = 0$. This completes the proof. □

PROOF OF PROPOSITION 5.2 (FOLLOWING [HARTLINE 2012]). We show the stronger statement that for every i, s_{-i}, $\mathbb{E}_{s_i}[p_i(\vec{s})] = \mathbb{E}_{s_i}[x_i(\vec{s})\varphi_i(s_i \mid s_{-i})] - (x_i(0, s_{-i})v_i(0, s_{-i}) - p_i(0, s_{-i}))$. The claim follows by linearity of expectation.

We fix i, s_{-i} and omit them from the notation for simplicity. Recall that the conditional revenue curve is

$$B(s) = v(s) \int_s^{\omega} f(t)dt.$$

Using the equation $p(s) = p(0) + \int_0^s x'(t)v(t)dt$ derived in the proof of Proposition 5.1, we get

$$\begin{aligned}
\mathbb{E}_s[p(s)] &= p(0) + \int_0^{\omega} f(s) \int_0^s x'(t)v(t)dtds \\
&= p(0) + \int_0^{\omega} x'(t)v(t) \int_t^{\omega} f(s)dsdt \\
&= p(0) + \int_0^{\omega} x'(t)B(t)dt \\
&= p(0) + x(t)B(t)|_0^{\omega} - \int_0^{\omega} x(t)B'(t)dt \\
&= p(0) - x(0)B(0) - \int_0^{\omega} x(t)\frac{B'(t)}{f(t)}f(t)dt \\
&= p(0) - x(0)v(0) - \int_0^{\omega} x(t)\varphi(t)f(t)dt \\
&= \mathbb{E}_s[x(s)\varphi(s)] - \left(x(0)v(0) - p(0)\right).
\end{aligned}$$

□