CPS 223

Game and Nash Equilibrium

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Nash's Proof and PPAD

(Slides borrowed from MIT **Topics in Algorithmic Game Theory** course by Constantinos Daskalakis)

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$

f: $[0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Proof

Kick / Dive	Left	Right
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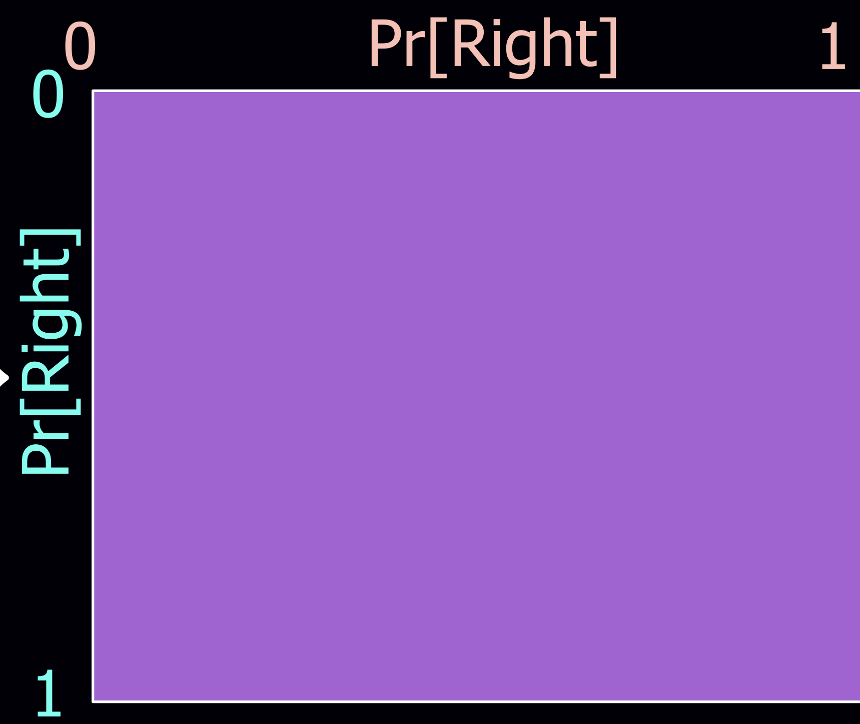
Penalty Shot Game

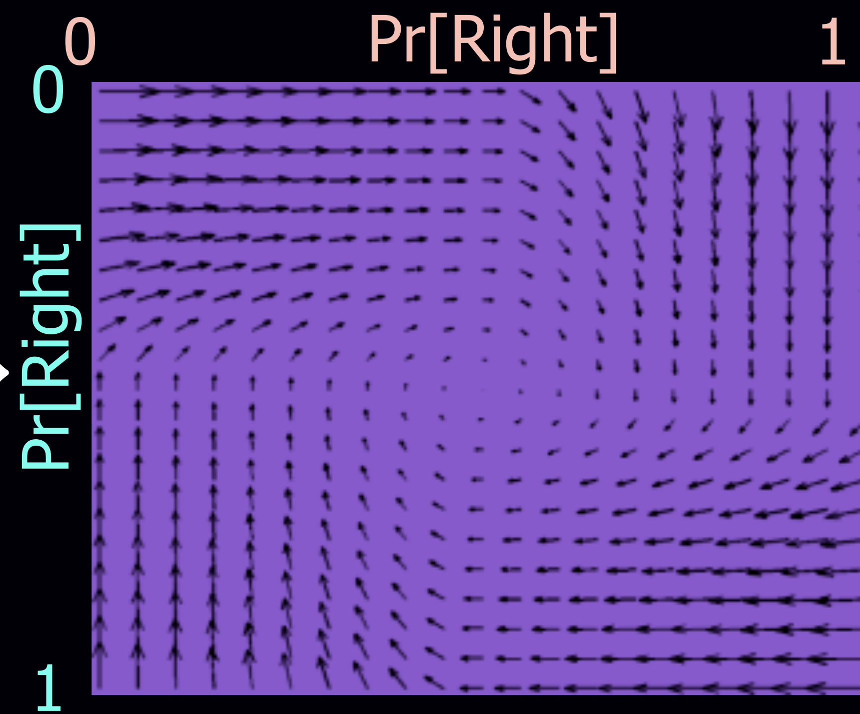


Visualizing Nash's Proof

Dive \ Kick	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$



Visualizing Nash's Proof

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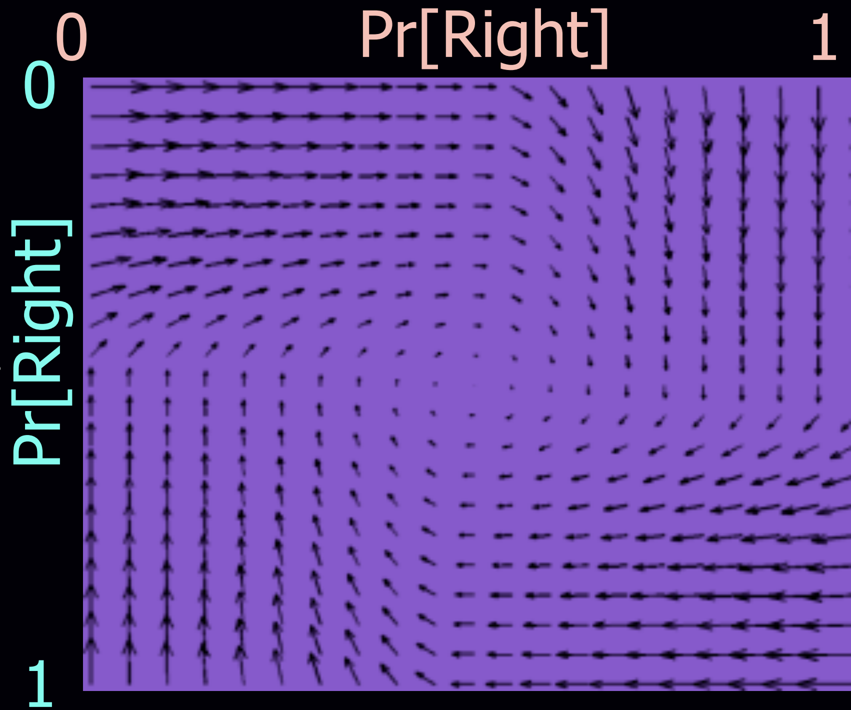
Penalty Shot Game

$\longmapsto$

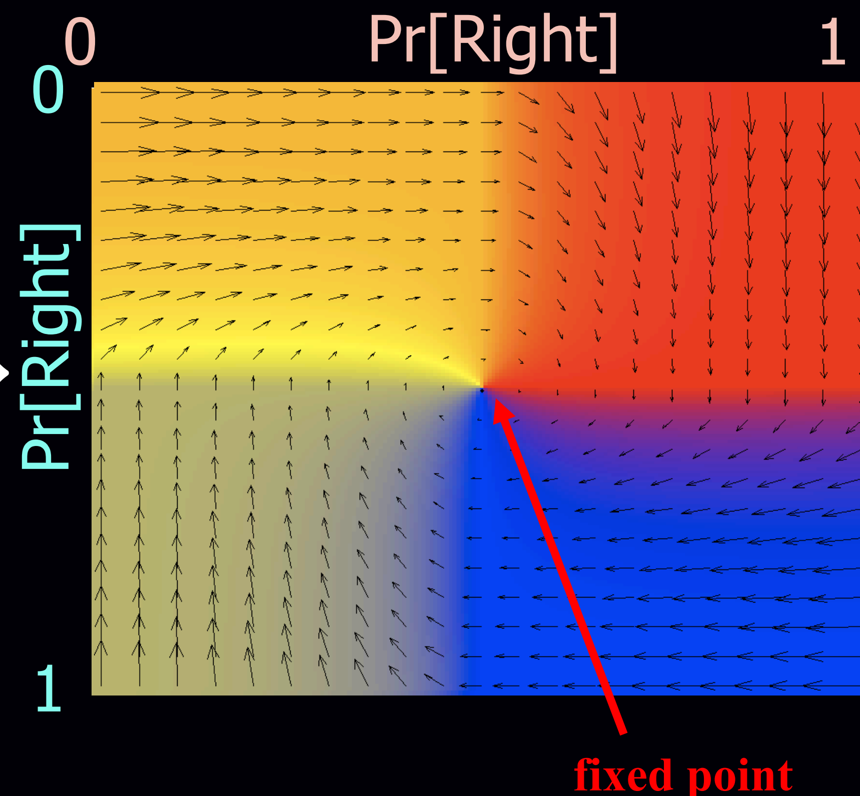


Visualizing Nash's Proof

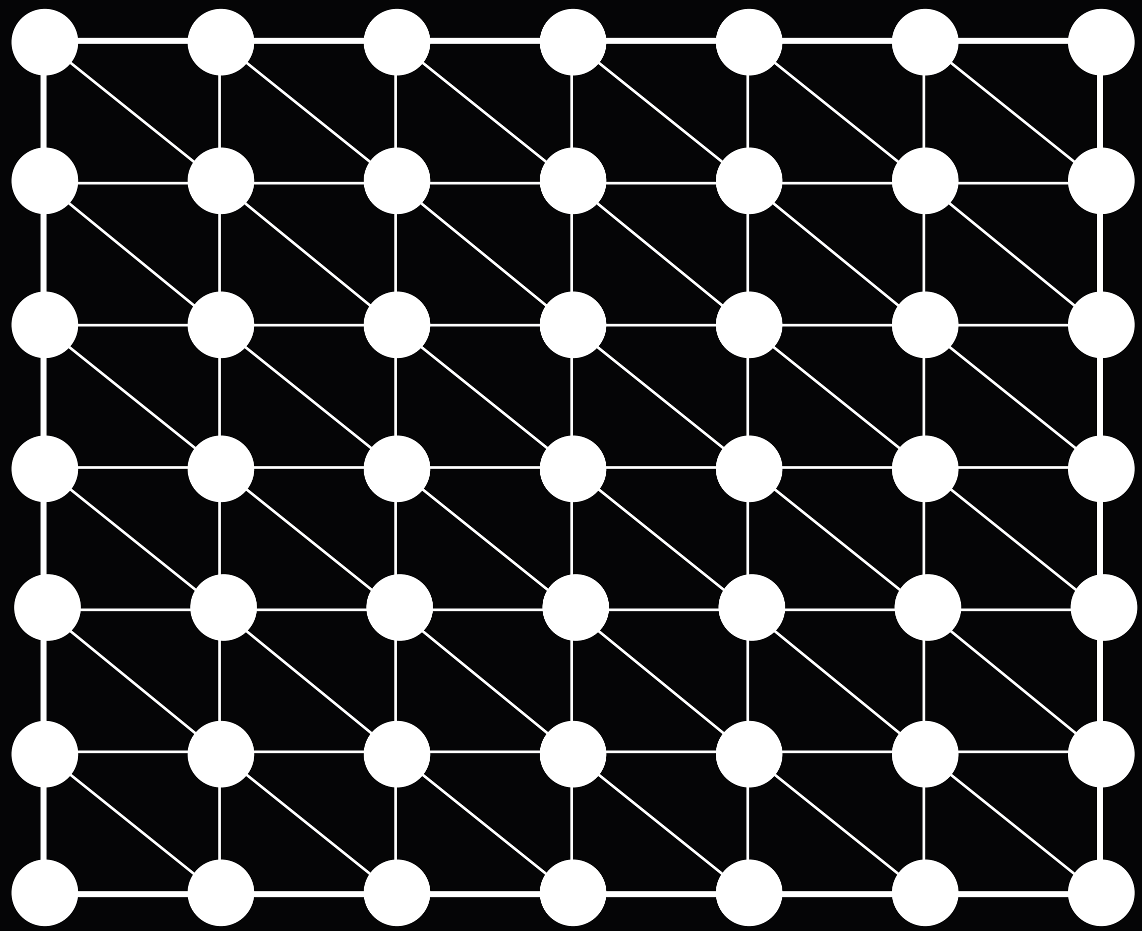
		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

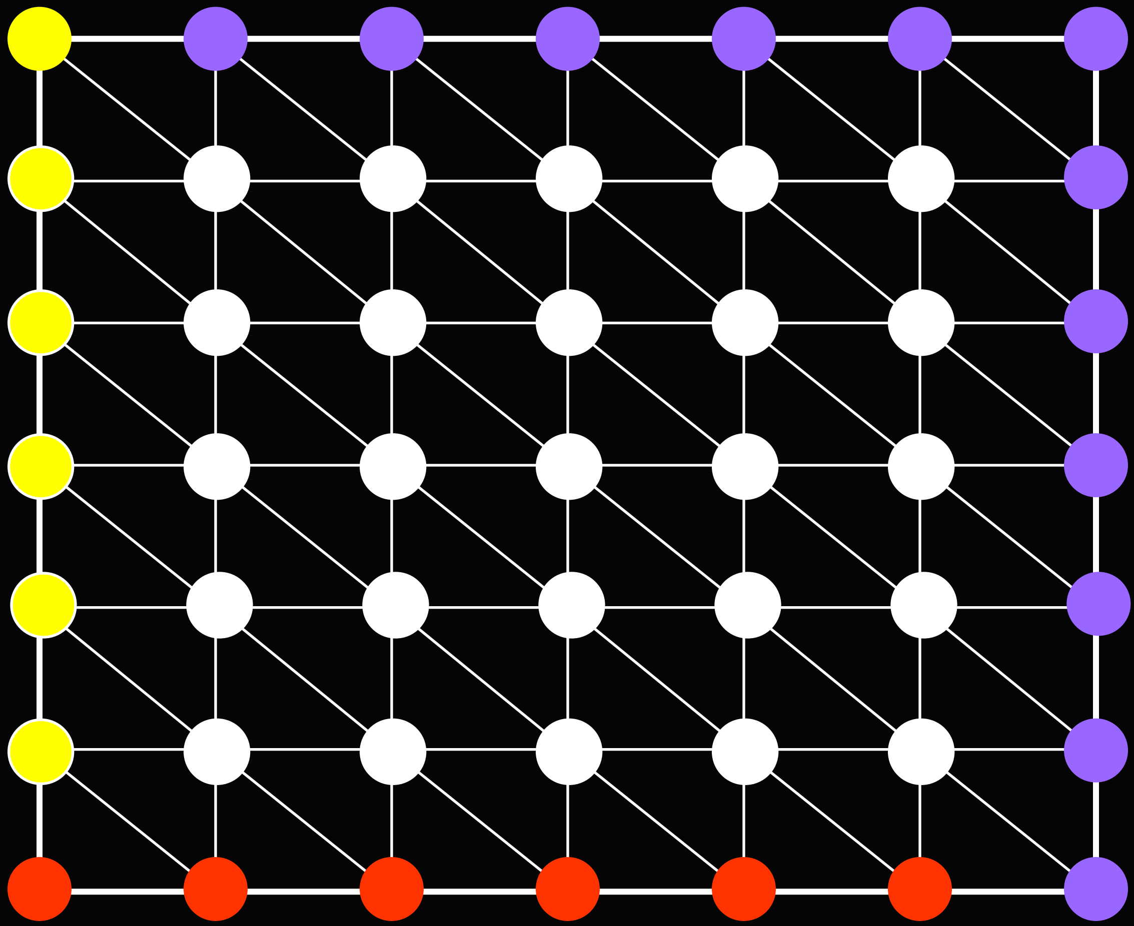
↦



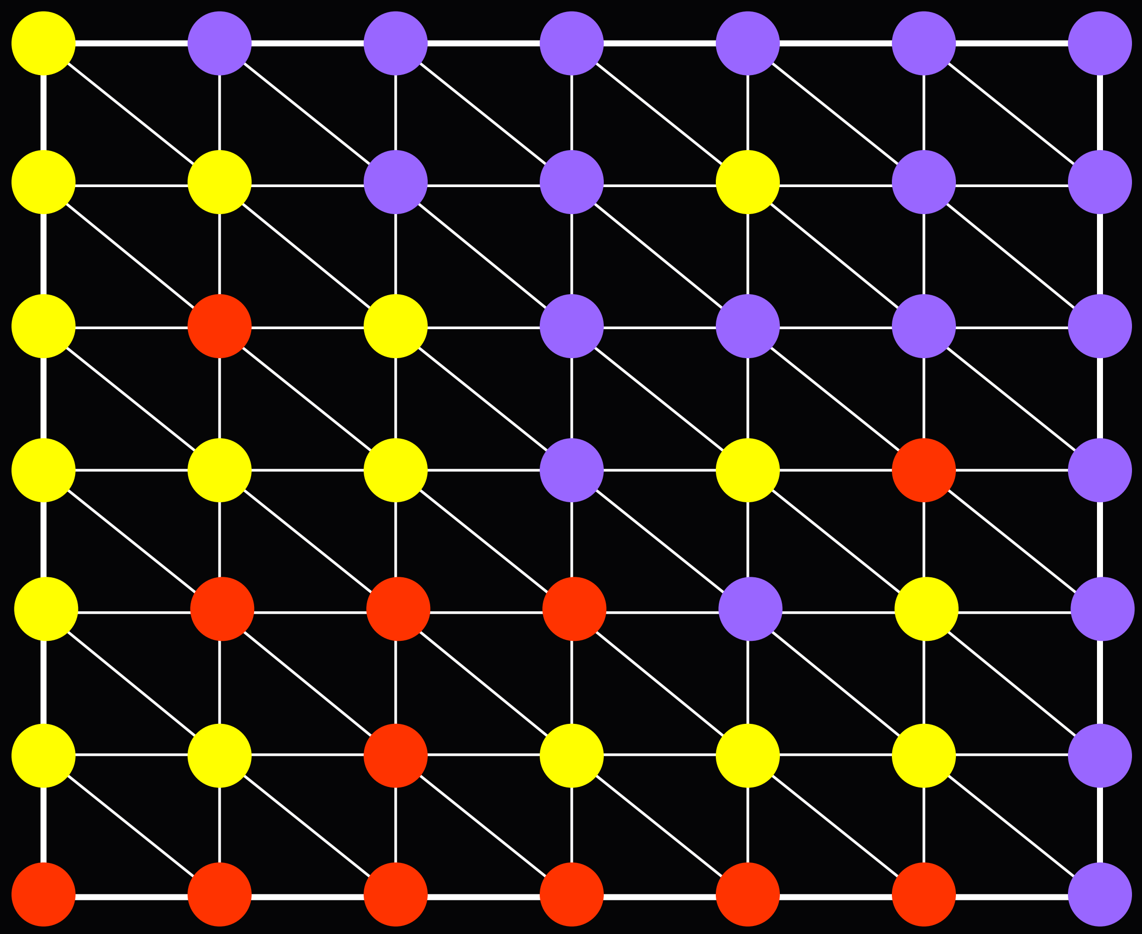
Sperner’s Lemma



Sperner's Lemma

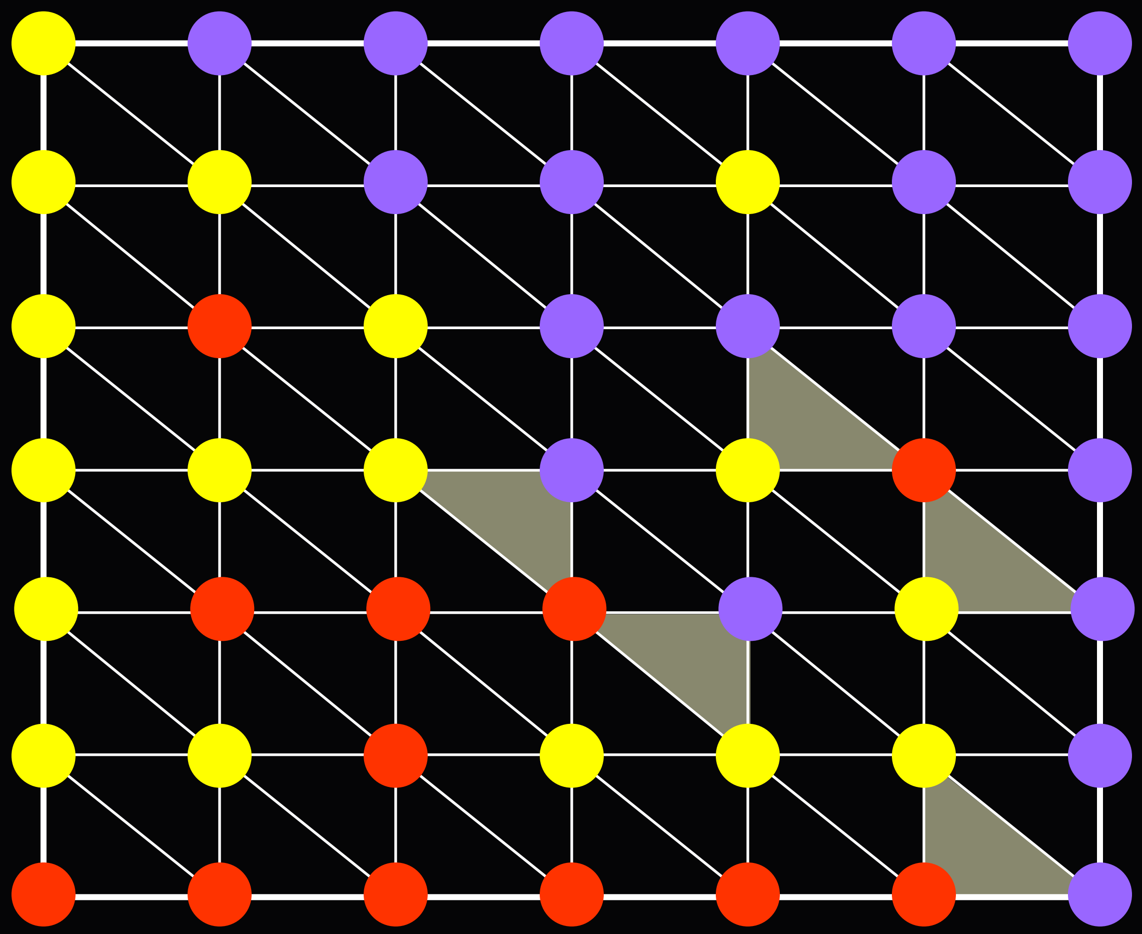


Sperner's Lemma



Lemma: No matter how the internal nodes are colored there exists a tri-chromatic triangle. In fact, an odd number of them.

Sperner's Lemma



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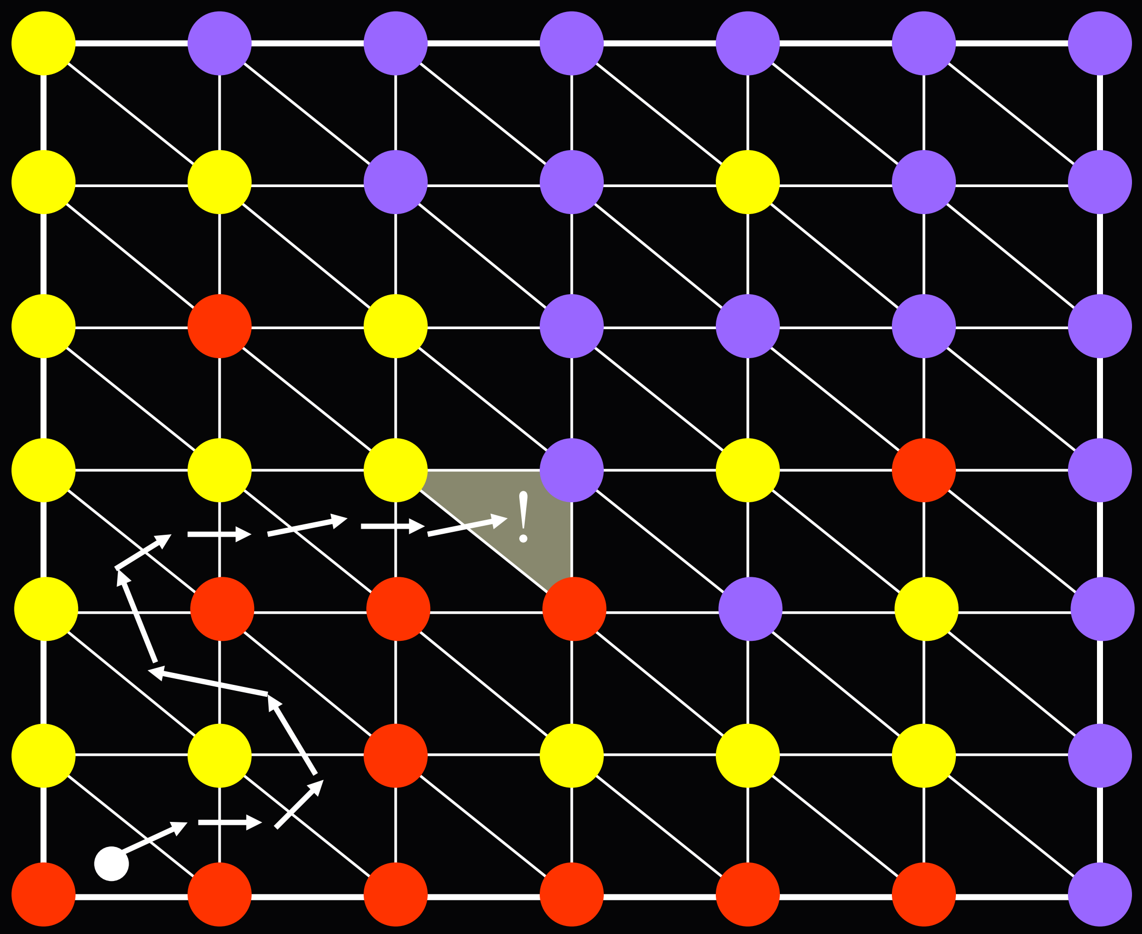

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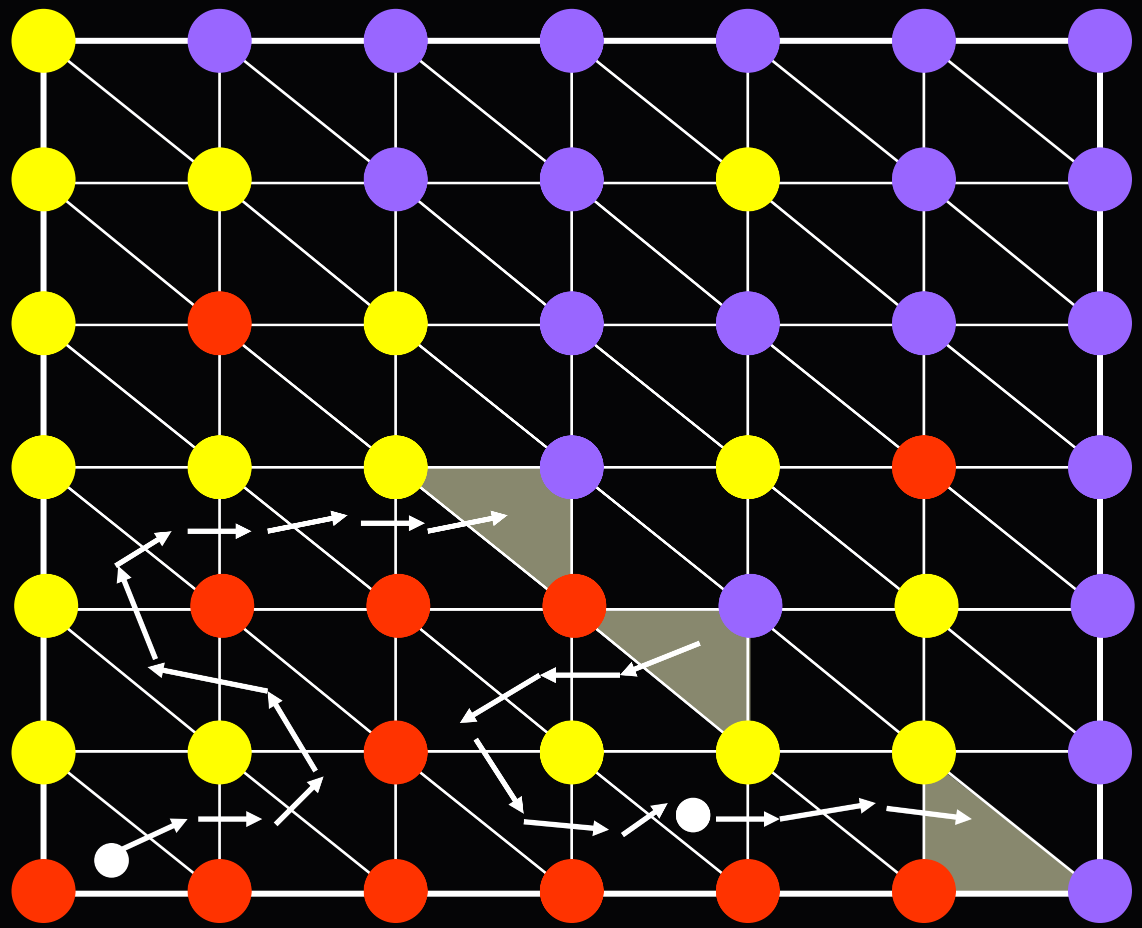

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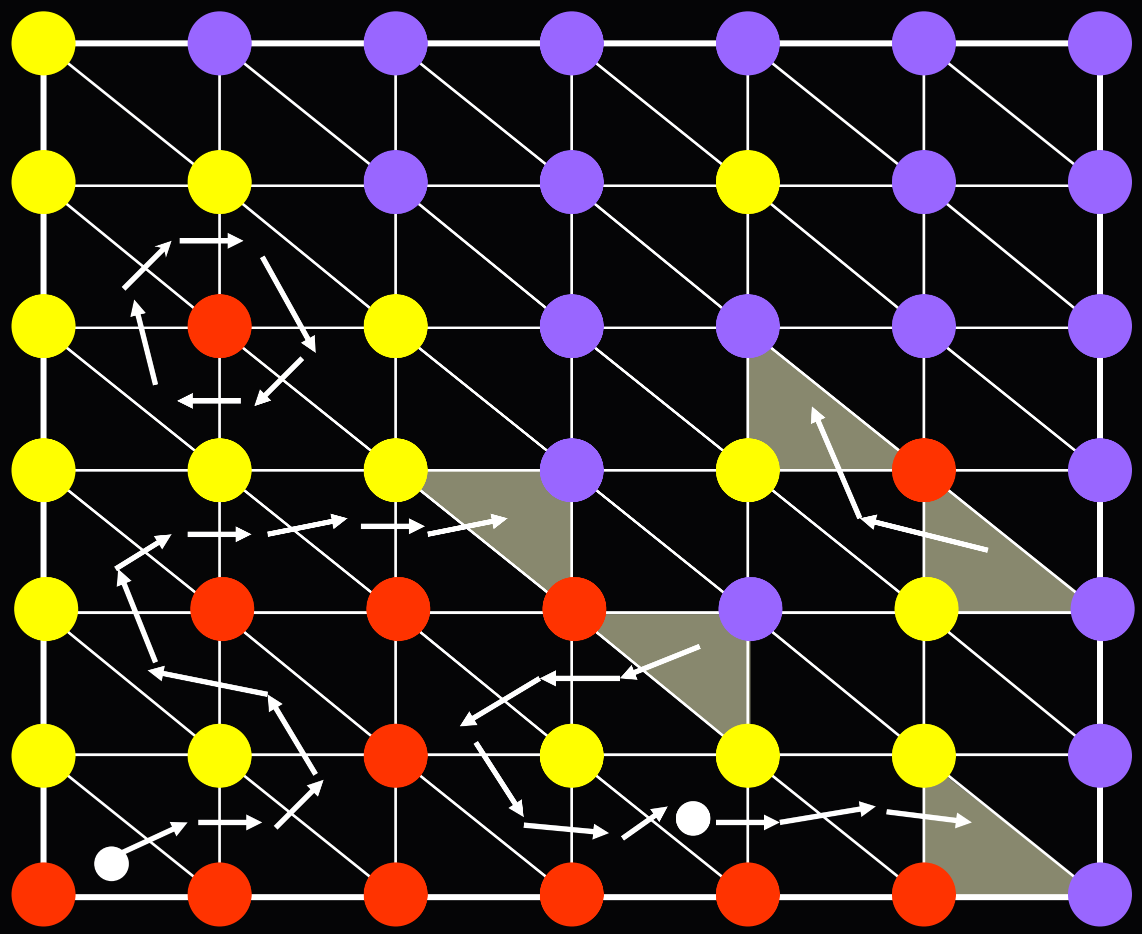
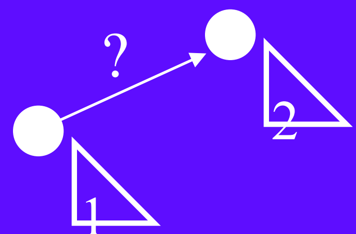


Lemma: No matter how the internal nodes are colored there exists a tri-chromatic triangle. In fact, an odd number of them.

Sperner's Lemma

Space of
Triangles

Transition Rule: *If ∃* ***red*** *-* ***yellow*** *door cross it with* ***yellow*** *on your left hand*



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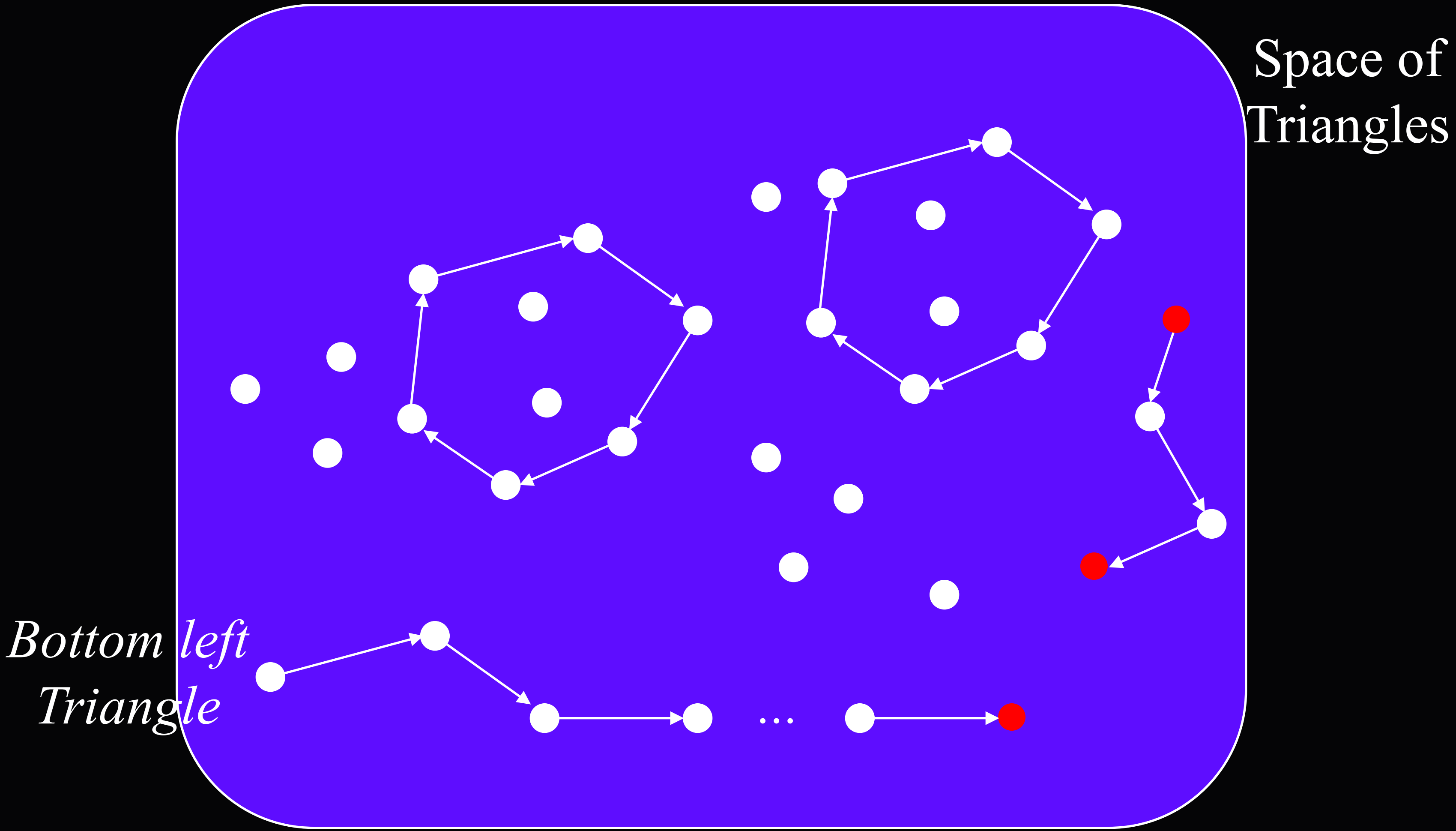
Sperner's Lemma



The PPAD Class [Papadimitriou'94]

The class of all problems with guaranteed solution by dint of the following graph-theoretic lemma

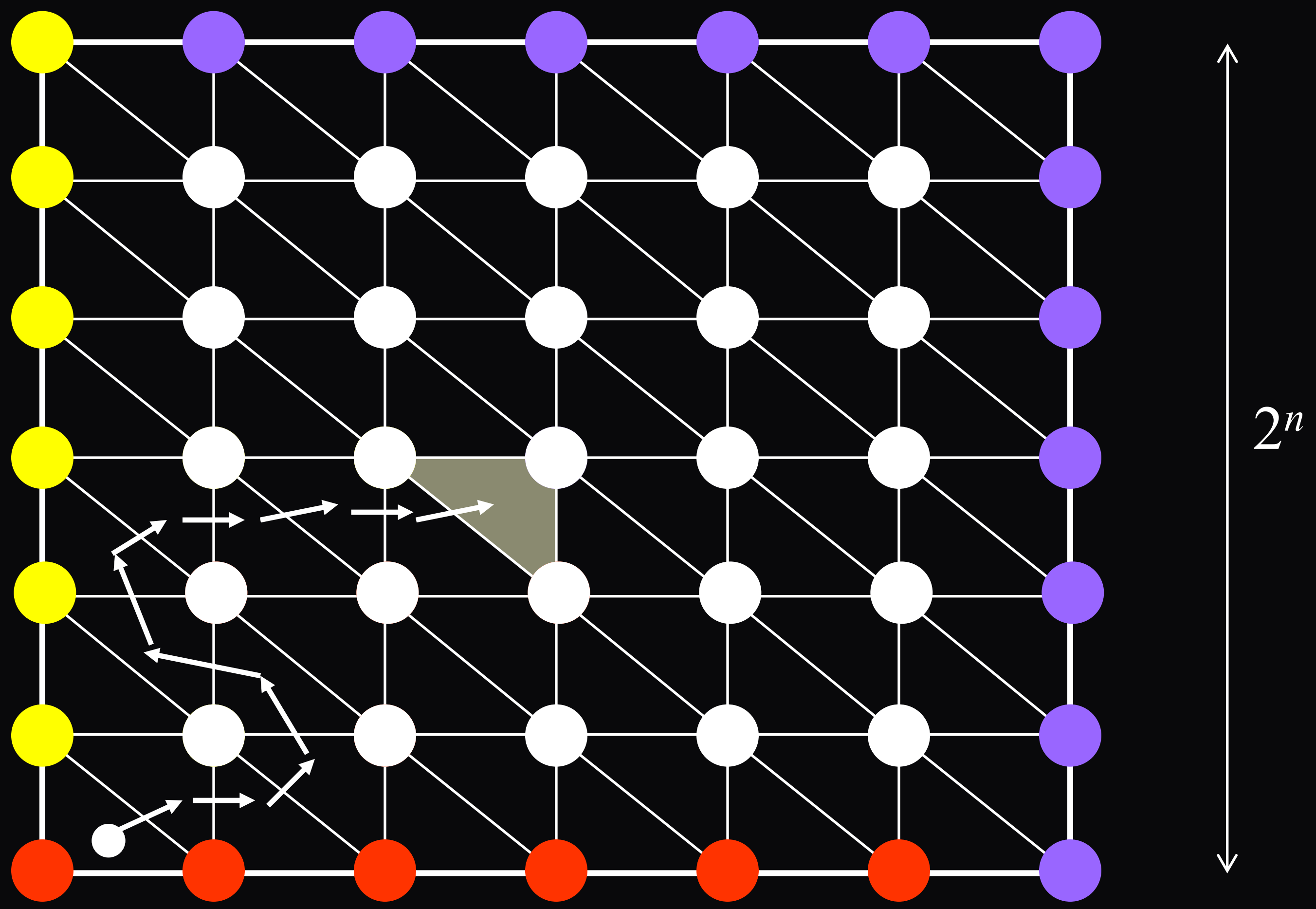
> *A directed graph with an unbalanced node (node with indegree $\neq$ outdegree) must have another.*

Such problems are defined by a directed graph *G*, and an unbalanced node *u* of *G; they require finding another unbalanced node.*

e.g. finding a Sperner triangle is in PPAD

But wait a second…given an unbalanced node in a directed graph, why is it not trivial to find another?

Solving SPERNER



However, the walk may wonder in the box for a long time, before locating the tri-chromatic triangle. Worst-case: 2^{2n}.

The PPAD Class

The class of all problems with guaranteed solution by dint of the following graph-theoretic lemma

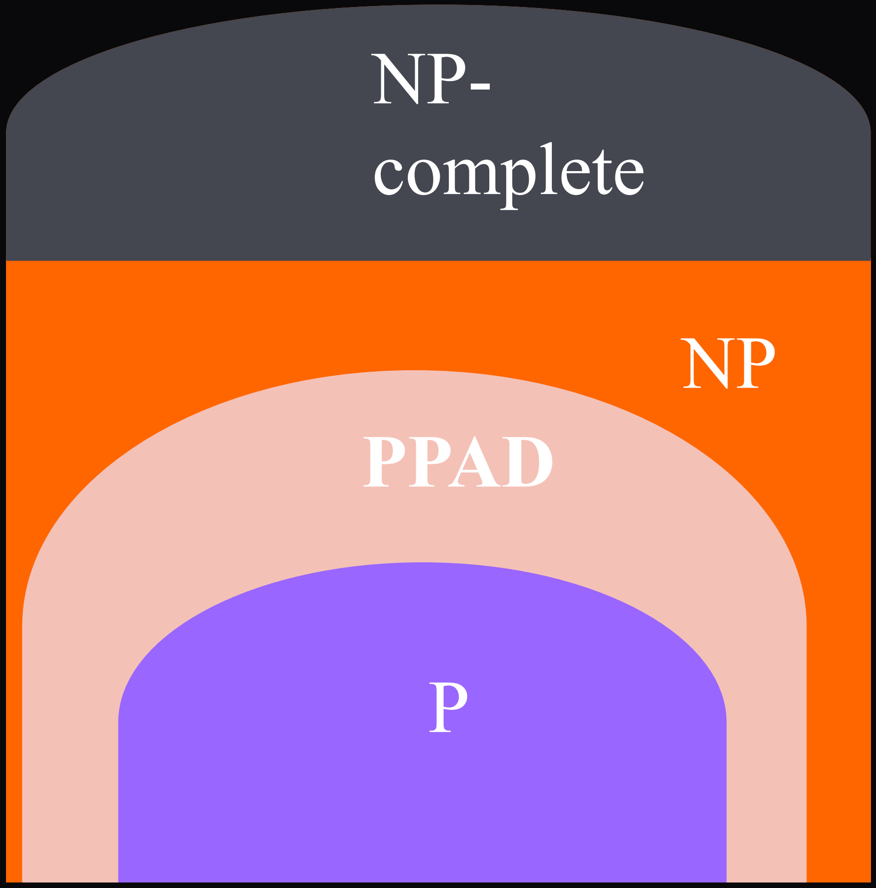
> *A directed graph with an unbalanced node (node with indegree $\neq$ outdegree) must have another.*

Such problems are defined by a directed graph *G* (huge but implicitly defined), and an unbalanced node *u* of *G*; *they require finding another unbalanced node.*

e.g. SPERNER $\in$ PPAD

Where is PPAD located w.r.t. NP?

(Believed) Location of PPAD



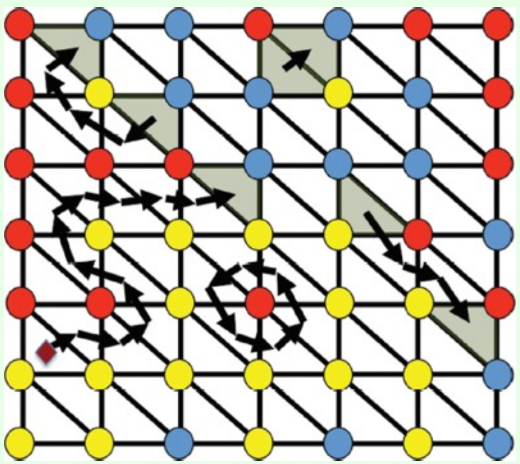
Finding Nash Equilibrium

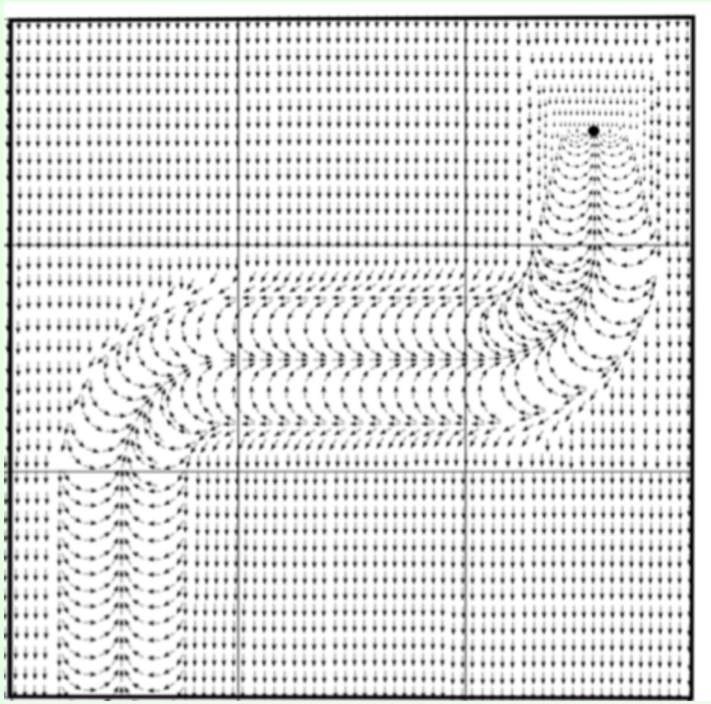
Games and Computation

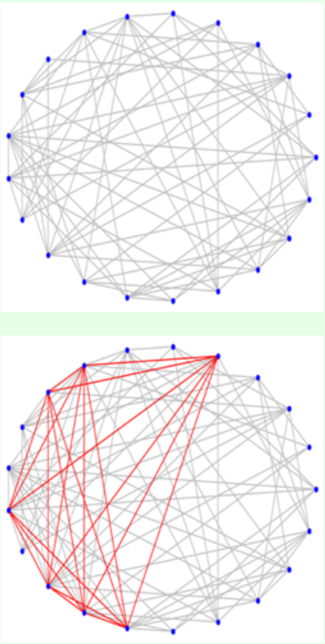
- [Nash 50] Every finite game has an equilibrium point
  - Finding it requires solving hard problems

Games and Computation

If one can find an (approximate) equilibrium







How hard is it to compute *one* (any) Nash equilibrium?

- Complexity was open for a long time
  - [Papadimitriou STOC01]: “together with factoring […] the most important concrete open question on the boundary of P today”
- Recent sequence of papers shows that computing one (any) Nash equilibrium is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in the worst case)

What if we want to compute a Nash equilibrium with a specific property?

- For example:
  - An equilibrium that is not Pareto-dominated
  - An equilibrium that maximizes the expected social welfare (= the sum of the agents' utilities)
  - An equilibrium that maximizes the expected utility of a given player
  - An equilibrium that maximizes the expected utility of the worst-off player
  - An equilibrium in which a given pure strategy is played with positive probability
  - An equilibrium in which a given pure strategy is played with zero probability
  - …
- All of these are NP-hard (and the optimization questions are inapproximable assuming P ≠ NP), even in 2-player games [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]

Search-based approaches (for 2 players)

- Suppose we know the support X_i of each player i's mixed strategy in equilibrium
  - That is, which pure strategies receive positive probability
- Then, we have a linear feasibility problem:
  - for both i, for any $s_i \in S_i - X_i$, $\mathbf{p_i(s_i)} = 0$
  - for both i, for any $s_i \in X_i$, $\Sigma \mathbf{p_{-i}(s_{-i})} u_i(s_i, s_{-i}) = \mathbf{u_i}$
  - for both i, for any $s_i \in S_i - X_i$, $\Sigma \mathbf{p_{-i}(s_{-i})} u_i(s_i, s_{-i}) \leq \mathbf{u_i}$
- Thus, we can search over possible supports
  - This is the basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]
- Dominated strategies can be eliminated

Solving for a Nash equilibrium using MIP (2 players)

[Sandholm, Gilpin, Conitzer AAAI05]

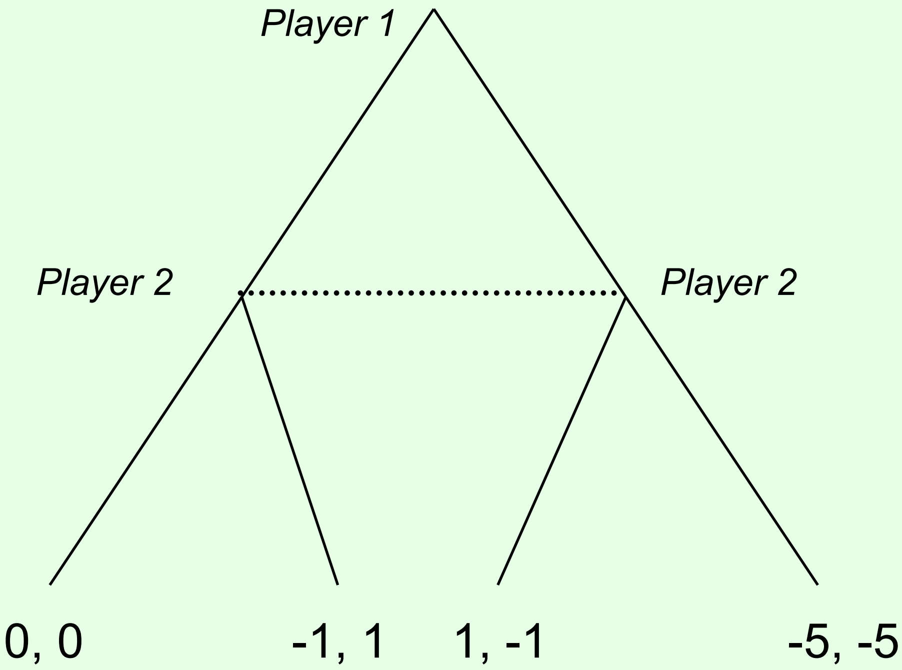
- maximize *whatever you like (e.g., social welfare)*
- subject to
  - for both i, for any s_i, $\Sigma_{s_{-i}} \mathbf{p}_{\mathbf{s}_{-i}} u_i(s_i, s_{-i}) = \mathbf{u}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{u}_i \geq \mathbf{u}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{p}_{\mathbf{s}_i} \leq \mathbf{b}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{u}_i - \mathbf{u}_{\mathbf{s}_i} \leq M(1 - \mathbf{b}_{\mathbf{s}_i})$
  - for both i, $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} = 1$

- $\mathbf{b}_{\mathbf{s}_i}$ is a binary variable indicating whether s_i is in the support, M is a large number

Extensive-Form Games

Imperfect information

- Dotted lines indicate that a player cannot distinguish between two (or more) states
  - A set of states that are connected by dotted lines is called an information set
- Reflected in the normal-form representation

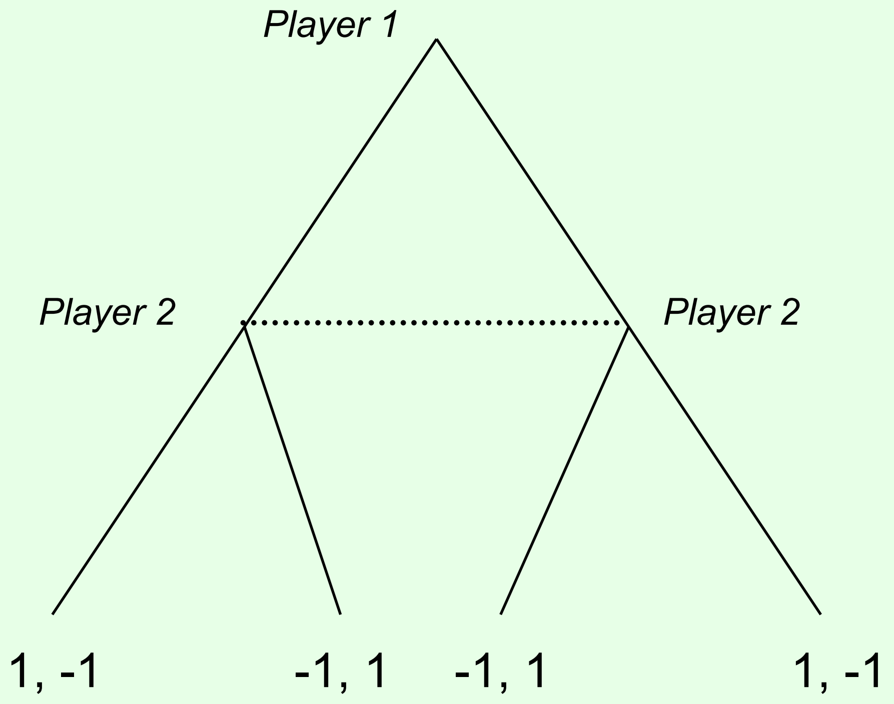


	L	R
L	0, 0	-1, 1
R	1, -1	-5, -5

- Any normal-form game can be transformed into an imperfect-information extensive-form game this way

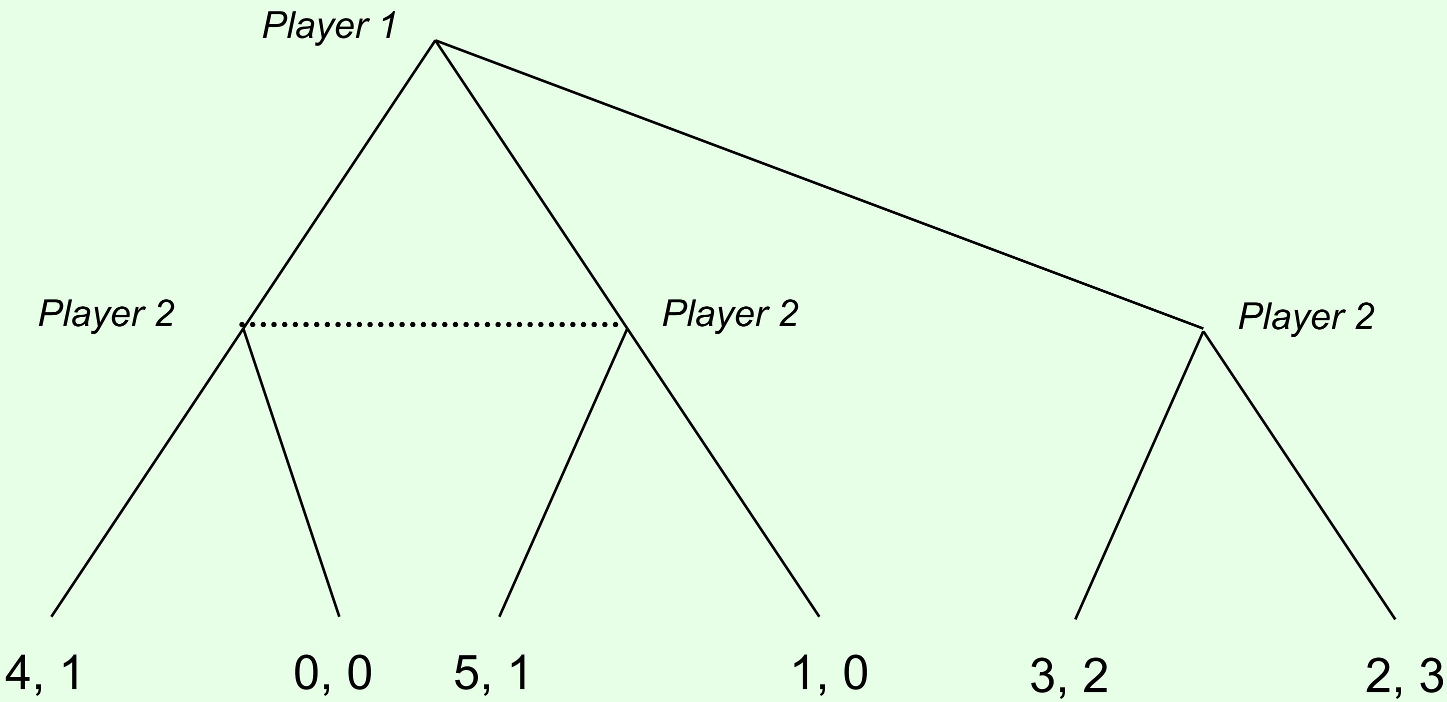
Subgame perfection and imperfect information

- How should we extend the notion of subgame perfection to games of imperfect information?



- We cannot expect Player 2 to play Right after Player 1 plays Left, and Left after Player 1 plays Right, because of the information set
- Let us say that a subtree is a subgame only if there are no information sets that connect the subtree to parts outside the subtree

Subgame perfection and imperfect information…



- One of the Nash equilibria is: (R, RR)
- Also subgame perfect (the only subgames are the whole game, and the subgame after Player 1 moves Right)
- But it is not reasonable to believe that Player 2 will move Right after Player 1 moves Left/Middle (not a credible threat)
- There exist more sophisticated refinements of Nash equilibrium that rule out such behavior

Computing equilibria in the extensive form

- Can just use normal-form representation
  - Misses issues of subgame perfection, etc.
- Another problem: there are exponentially many pure strategies, so normal form is exponentially larger
  - Even given polynomial-time algorithms for normal form, time would still be exponential in the size of the extensive form
- There are other techniques that reason directly over the extensive form and scale much better
  - E.g., using the sequence form of the game

Commitment

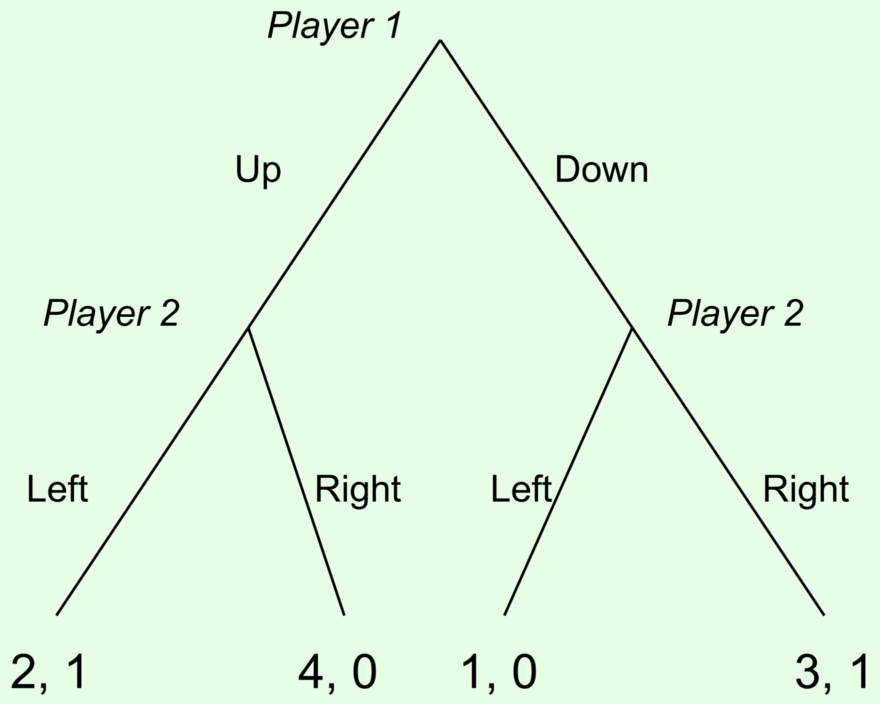
- Consider the following (normal-form) game:

2, 1	4, 0
1, 0	3, 1

- How should this game be played?
- Now suppose the game is played as follows:
  – Player 1 commits to playing one of the rows,
  – Player 2 observes the commitment and then chooses a column
- What is the optimal strategy for player 1?
- What if 1 can commit to a mixed strategy?

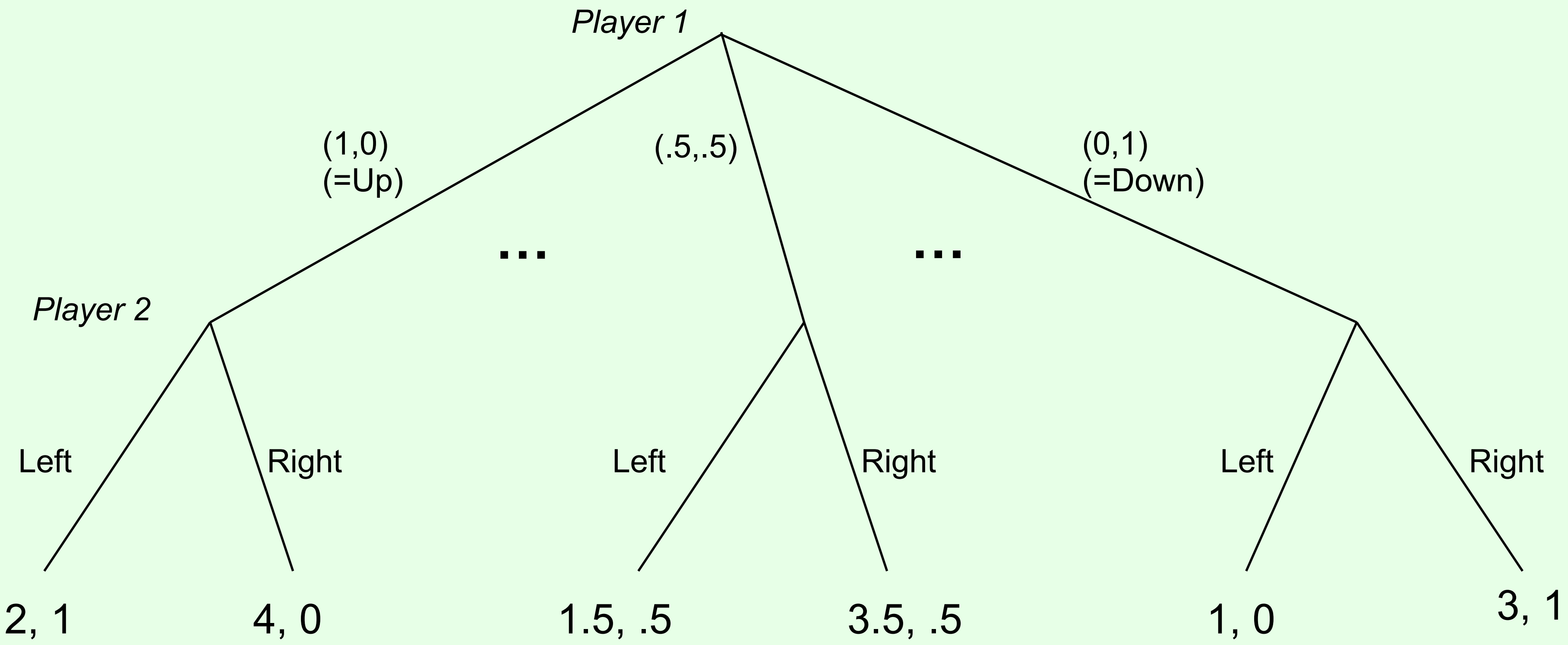
Commitment as an extensive-form game

- For the case of committing to a pure strategy:



Commitment as an extensive-form game

- For the case of committing to a mixed strategy:



- Infinite-size game; computationally impractical to reason with the extensive form here

Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

- For every column t separately, we will solve separately for the best mixed row strategy (defined by $\mathbf{p_s}$) that induces player 2 to play t
- maximize $\Sigma_s \mathbf{p_s} u_1(s, t)$
- subject to

  for any t', $\Sigma_s \mathbf{p_s} u_2(s, t) \geq \Sigma_s \mathbf{p_s} u_2(s, t')$

  $\Sigma_s \mathbf{p_s} = 1$
- (May be infeasible, e.g., if t is strictly dominated)
- Pick the t that is best for player 1

Visualization

	L	C	R
U	0,1	1,0	0,0
M	4,0	0,1	0,0
D	0,0	1,0	1,1