1 Overview

In Lecture 13, we introduced the notion of perfect matchings in bipartite graphs. We also saw Hall’s Theorem, which tells us when a bipartite graph has a perfect matching. In this lecture, we give a proof of Hall’s Theorem.

2 Hall’s Theorem

In this section, we re-state and prove Hall’s theorem. Recall that in a bipartite graph \( G = (A \cup B, E) \), an \( A \)-perfect matching is a subset of \( E \) that matches every vertex of \( A \) to exactly one vertex of \( B \), and doesn’t match any vertex of \( B \) more than once.

**Theorem 1** (Hall 1935). A bipartite graph \( G = (A \cup B, E) \) has an \( A \)-perfect matching if and only if the following condition holds:

\[
\forall S \subseteq A. |N(S)| \geq |S|,
\]

where \( N(S) = \{ v \in B : \exists u \in S. \{u, v\} \in E \} \).

**Remark 1:** If \( |A| = |B| \), then a matching is \( A \)-perfect if and only if it is perfect. So, in this case, Hall’s theorem tells us when a perfect matching exists. On the other hand, if \( |A| \neq |B| \), then \( G \) cannot contain a perfect matching because every edge of a matching pairs one vertex of \( A \) with exactly one vertex of \( B \). However, if \( |A| < |B| \), then \( G \) may still contain an \( A \)-perfect matching.

**Remark 2:** Theorem 1 is of the form \( P \leftrightarrow Q \), where \( P \) is the proposition “\( G \) has an \( A \)-perfect matching” and \( Q \) is known as Hall’s condition. In general, \( P \rightarrow Q \) states that \( Q \) is a necessary condition for \( P \). (To see why, consider the contrapositive \( \neg Q \rightarrow \neg P \).) Furthermore, \( Q \rightarrow P \) states that \( Q \) is a sufficient condition for \( P \). Thus, Hall’s theorem states that Hall’s condition is a necessary and sufficient condition for a bipartite graph to have an \( A \)-perfect matching.

**Proof.** We now begin the proof of Theorem 1.

**Hall’s condition is necessary:** Assume that \( G \) has an \( A \)-perfect matching, which we denote by \( M \). Let \( S \) be an arbitrary subset of \( A \). Since \( M \) is an \( A \)-perfect matching, \( M \) matches every vertex of \( S \) to exactly one vertex of \( B \), and no vertex of \( B \) is matched more than once. So if we restrict \( G \) to the edges in \( M \), the vertices of \( S \) each have a distinct neighbor in \( N(S) \). Since \( N(S) \) is defined using all the edges of \( G \) and \( M \) is only a subset of \( E \), this implies \( |N(S)| \geq |S| \).

**Hall’s condition is sufficient:** We will construct an \( A \)-perfect matching \( M \) by proceeding with induction on \( |A| \), assuming \( G \) satisfies Hall’s condition.

**Base case:** \( |A| = 1 \). Let \( a \) denote the sole vertex of \( A \). Hall’s condition tells us \( |N(\{a\})| \geq 1 \), which means \( a \) has at least one neighbor. We can set \( M = \{\{a, b\}\} \) where \( b \) is any neighbor of \( a \).
Then, $M$ is an $A$-perfect matching: every vertex of $A$ is matched, and no vertex of $B$ is matched more than once.

**Inductive hypothesis (IH):** Assume that for all $k$ such that $1 \leq k \leq |A| - 1$, any bipartite graph $H = (C \cup D, F)$ satisfying $|C| = k$ has a $C$-perfect matching if and only if $H$ satisfies Hall’s condition on $C$, i.e., $\forall S \subseteq C. |N(S)| \geq |S|$.

**Inductive step:** We will now construct an $A$-perfect matching $M$ in $G$, starting with $M = \emptyset$. Note that when $G$ satisfies Hall’s condition, there are two possible cases: the inequality is strict for every $S$ that is a strict subset of $A$ (i.e., $\forall S \subset A. |N(S)| > |S|$), or there exists at least one $S \subset A$ such that $|N(S)| = |S|$.

1. **In the first case**, since $|N(S)|$ and $|S|$ are integers, we can assume
   $$\forall S \subset A. |N(S)| \geq |S| + 1.$$  
   (1)

   We claim that the following procedure returns a perfect matching:
   
   1. Let $u$ be an arbitrary vertex of $A$, and add any edge $e = \{u, v\}$ of $E$ to $M$.
   2. Remove $u, v$, and all edges incident to $u$ or $v$ from $G$ to construct graph $G'$.
   3. By the IH, $G'$ has a matching $M'$ that is $(A \setminus \{u\})$-perfect.
   4. Add the edges of $M'$ to $M$, and return $M$.

   ![Figure 1: The original graph $G$ with an arbitrary edge $\{u, v\}$ added to $M$ (bold). The graph $G'$ comprises the remaining vertices.](image1)

   ![Figure 2: Applying the IH on $G'$ yields a matching $M'$ that is $(A \setminus \{u\})$-perfect. The edges of the final matching $M$ are in bold.](image2)

   For this procedure to be correct, we have to show several properties. First, to show that Step 1 is valid, we need to establish that the degree of any vertex $u \in A$ is at least 1. If this does not hold, then $N(\{u\}) = 0$, while $|\{u\}| = 1$, thereby violating Hall’s condition for $S = \{u\}$.

   Next, we show that $G'$ satisfies Hall’s condition so that Step 3 is valid. Let $S'$ be any subset of $A \setminus \{u\}$, and let $N'(S')$ denote its neighbors in $G'$ (so $N'(S') \subseteq B \setminus \{v\}$). Notice that $|N'(S')| - 1 \geq |S'|$ because of (1). Since we only removed one vertex of $B$ to construct $G'$, $|N'(S')| \geq |N(S')| - 1$. Taken together, these inequalities imply $|N'(S')| \geq |S'|$, as desired.

   Now we must show that $M = M' \cup \{u, v\}$ is an $A$-perfect matching. Since $M'$ matches every vertex of $A \setminus \{u\}$ and $e$ matches $u$, every vertex of $A$ is indeed matched by $M$. Furthermore,
vertex \( v \) is matched once because \( G' \) excludes \( v \), and the vertices of \( B \setminus \{ v \} \) are matched at most once because \( M' \) is a matching. Thus, \( M \) is an \( A \)-perfect matching.

(ii) In the second case, we assume there exists \( S \subset A \) such that \(|N(S)| = |S|\). We claim that the following procedure returns a perfect matching:

1. Partition \( A \) into \( S \) and \( \overline{S} = A \setminus S \) and \( B \) into \( N(S) \) and \( \overline{N(S)} = B \setminus N(S) \).
2. Let \( G_1 = (S \cup N(S), E_1) \) where \( E_1 \) denotes the edges of \( G \) among \( S \cup N(S) \). By the IH, \( G_1 \) has a matching \( M_1 \) that is \( S \)-perfect.
3. Let \( G_2 = (\overline{S} \cup \overline{N(S)}, E_2) \), where \( E_2 \) denotes the edges of \( G \) among \( \overline{S} \cup \overline{N(S)} \). By the IH, \( G_2 \) has a matching \( M_2 \) that is \( \overline{S} \)-perfect.
4. Let \( M = M_1 \cup M_2 \), and return \( M \).

As in the previous case, we must prove that \( G_1 \) and \( G_2 \) satisfy Hall’s condition so that Step 2 and Step 3 are valid. The graph \( G_1 \) is easier to handle: observe that there are no edges from \( S \) to \( \overline{N(S)} \). Thus, for any subset \( T \) of \( S \), its neighborhood in \( G_1 \) is exactly the same as its neighborhood in \( G \). Since \( G \) satisfies Hall’s condition, this implies \( G_1 \) also satisfies Hall’s condition.

Showing that \( G_2 \) also satisfies Hall’s condition is slightly trickier. For contradiction, suppose \( G_2 \) violates Hall’s condition. This means there exists \( X \subseteq S \) such that \(|Y| < |X|\), where \( Y \) denotes the neighborhood of \( X \) in \( G_2 \), i.e., \( Y = N(X) \cap \overline{N(S)} \). Now consider the set \( S \cup X \). This is a subset of \( A \), so since \( G \) satisfies Hall’s condition, we know that

\[
|N(S \cup X)| \geq |S \cup X|.
\]

Furthermore, since \( S \) and \( X \) are disjoint, \(|S \cup X| = |S| + |X|\). Also, notice that the only neighbors of \( S \cup X \) contained in \( \overline{N(S)} \) are the neighbors of \( X \), i.e., \( N(S \cup X) = N(S) \cup Y \). Finally, since \( N(S) \) and \( Y \) are disjoint, we know that \(|N(S) \cup Y| = |N(S)| + |Y|\). Putting this all together, we get

\[
|N(S \cup X)| = |N(S) \cup Y| = |N(S)| + |Y| = |S| + |Y| < |S| + |X| = |S \cup X|.
\]

\( (\text{definitions of } S, X, Y) \)

\( (\text{\( N(S) \) and \( Y \) are disjoint}) \)

\( (\text{\( \text{defining property of } S \))} \)

\( (\text{\( \text{defining property of } X \))} \)

\( (\text{\( S \text{ and } X \) are disjoint}) \)
Thus, the set \( S \cup X \) violates Hall’s condition in the original graph \( G \) because \(|N(S \cup X)| < |S \cup X|\). This concludes the proof that \( G_2 \) satisfies Hall’s condition.

Now that we know \( G_1 \) and \( G_2 \) satisfy Hall’s condition, we must show that \( M = M_1 \cup M_2 \) is an \( A \)-perfect matching. Since \( M_1 \) is \( S \)-perfect and \( M_2 \) is \( \overline{S} \)-perfect, we know that \( M \) matches every vertex of \( S \cup \overline{S} = A \). Furthermore, no edges of \( M_1 \) and \( M_2 \) share an endpoint because \( M_1 \) and \( M_2 \) were obtained from two disjoint graphs \( G_1 \) and \( G_2 \). Thus, no vertex of \( B \) is matched twice by \( M \), so \( M \) is an \( A \)-perfect matching.

\[ \square \]

3 Summary

In this lecture, we proved Hall’s theorem, one of the most well-known results in discrete mathematics. The proof uses induction in a manner that is more complicated than typical induction proofs we have seen.