1 Overview

In this lecture, we continue studying relations and functions.

2 Relations and Functions

Recall that a relation is defined between two sets as a subset of the Cartesian product: \( R \subseteq A \times B \) for sets \( A, B \). We use the following shorthand:

- \( A \text{ bij } B \) ("\( A \) has a bijection with \( B \)") if there exists a bijective total function \( R \subseteq A \times B \).
- \( A \text{ surj } B \) ("\( A \) has a surjection with \( B \)") if there exists a surjective total function \( R \subseteq A \times B \).
- \( A \text{ inj } B \) ("\( A \) has an injection with \( B \)") if there exists an injective total function \( R \subseteq A \times B \).

Suppose \( A \) and \( B \) are finite sets. Then we have the following cardinality rules:

1. \( A \text{ bij } B \) \( \iff \) \( |A| = |B| \)
2. \( A \text{ surj } B \) \( \iff \) \( |A| \geq |B| \)
3. \( A \text{ inj } B \) \( \iff \) \( |A| \leq |B| \)

Proof of Rule 1. Suppose \( A \text{ bij } B \). This means there exists \( R \subseteq A \times B \) such that \( R \) is a bijective total function. Since \( R \) is a total function, every element of \( A \) maps to exactly one element of \( B \). Thus, the total sum of in-degrees of elements of \( B \) is \( |A| \). Since \( R \) is a bijection, every element \( b \in B \) is mapped to exactly once. In other words, each \( b \in B \) has in-degree one. Equivalently, for all \( b \in B \), \( \text{in-degree}(b) = 1 \). We have the following:

\[
|A| = \sum_{b \in B} \text{in-degree}(b) = \sum_{b \in B} 1 = |B|
\]

Thus, \( |A| = |B| \) as desired.

Proof of Rule 2. Suppose \( A \text{ surj } B \). This means there exists \( R \subseteq A \times B \) such that \( R \) is a surjective total function. Since \( R \) is a total function, every element of \( A \) maps to exactly one element of \( B \). Thus, the total sum of in-degrees of elements of \( B \) is \( |A| \). Since \( R \) is a surjection, every element in \( B \) is either mapped to once or more than once. Equivalently, for all \( b \in B \), \( \text{in-degree}(b) \geq 1 \). We have the following:

\[
|A| = \sum_{b \in B} \text{in-degree}(b) \geq \sum_{b \in B} 1 = |B|
\]

Thus, \( |A| \geq |B| \) as desired.
Proof of Rule 3. Suppose $A \text{ inj } B$. This means there exists $R \subseteq A \times B$ such that $R$ is a injective total function. Since $R$ is a total function, every element of $A$ maps to exactly one element of $B$. Thus, the total sum of in-degrees of elements of $B$ is $|A|$. Since $R$ is a injection, every element in $B$ is either mapped once or not at all. Equivalently, for all $b \in B$, $\text{in-degree}(b) \leq 1$. We have the following:

$$|A| = \sum_{b \in B} \text{in-degree}(b) \leq \sum_{b \in B} 1 = |B|$$

Thus, $|A| \leq |B|$ as desired. \qed

Definition 1. The inverse relation, denoted $R^{-1}$, of a relation $R$ is the set of ordered pairs obtained by reversing those of $R$. If $R \subseteq A \times B$:

$$aR^{-1}b \iff bRa.$$ 

Thus, $R^{-1} \subseteq B \times A$. Informally, $R^{-1}$ is the relation obtained by changing the direction of arrows in the mapping diagram of $R$.

If $R$ is a bijective total function, then $R^{-1}$ is also a bijective total function. All out-degrees and in-degrees of $R$ are exactly 1, so in $R^{-1}$ the out-degrees become in-degrees, and vice versa. This leads to the following theorem:

Theorem 1. $A \text{ bij } B$ if and only if $B \text{ bij } A$.

Proof. Suppose $A \text{ bij } B$. By the cardinality rules, $A \text{ bij } B$ if and only if $|A| = |B|$. Equivalently, $|B| = |A|$. By the cardinality rules, $|B| = |A|$ if and only if $B \text{ bij } A$. In addition to existence, if we have a bijection from $A$ to $B$, we can find a bijection from $B$ to $A$. Let $R$ be a bijective total function from $A$ to $B$. Taking the inverse of $R$ results in a bijective total function from $B$ to $A$, as previously observed. \qed

We present some observations when $A$, $B$ are finite sets.

Observation 2. If $A \text{ surj } B$ and $B \text{ surj } A$, then $A \text{ bij } B$.

This follows from the cardinality rules: $A \text{ surj } B$ implies $|A| \geq |B|$ and $B \text{ surj } A$ implies $|B| \geq |A|$. Together, these imply $|A| = |B|$. From our cardinality rules we know $|A| = |B|$ if and only if $A \text{ bij } B$.

Observation 3. Either $A \text{ surj } B$ or $B \text{ surj } A$.

This follows from the cardinality rules. For any finite sets $A$ and $B$, either $|A| \leq |B|$ or $|B| \leq |A|$. In the first case, $|A| \leq |B|$ implies $B \text{ surj } A$ and in the latter case $|B| \leq |A|$ implies $A \text{ surj } B$.

Observation 4. $A \text{ surj } B$ if and only if $B \text{ inj } A$.

Again, we use the cardinality rules. $A \text{ surj } B$ if and only if $|A| \geq |B|$. We know $|B| \leq |A|$ if and only if $B \text{ inj } A$. 


3 Properties of Relations on a Set

We say $R$ is defined on set $A$ if $R \subseteq A \times A$.

Example 1: Consider the set of positive integers, $\mathbb{Z}^+$. We can use comparators (such as $<$, $\leq$, $=$, $\geq$, $>$) to define relations on this set. Consider defining a relation using $\leq$. This means $(a, b) \in R \iff a \leq b$. For example, the pair $(3, 5) \in R$ since $3 \leq 5$, but $(5, 3) \notin R$ because $5 \not\leq 3$.

Example 2: Consider relation $R$ defined by $<$ on $\mathbb{Z}^+$. Total? Yes, $\forall x \in \mathbb{Z}^+ x < x + 1 \Rightarrow (x, x + 1) \in R$.

Function? No, $3 < 4, 3 < 5, 3 < 6$, etc. An out-degree of an element could be larger than 1.

Injective? No, $1 < 5, 2 < 5, 3 < 5, 4 < 5$. The in-degree of an element could be larger than 1.

Surjective? No, there is no positive integer less than 1 so $(x, 1) \notin R$.

We introduce new properties of relations defined when the domain and codomain are the same set.

Definition 2. A relation $R$ on set $A$ is reflexive when every element is related to itself. Formally,

$$aRa \ \forall a \in A.$$ 

Definition 3. A relation $R$ on set $A$ is symmetric when $a$ relates to $b$ if and only if $b$ relates to $a$. In other words,

$$aRb \iff bRa \ \forall a, b \in A.$$ 

Definition 4. A relation $R$ on set $A$ is transitive if for every pair $(a, b)$ if $b$ also relates to some element $c$, then $a$ must also relate to $c$. Formally,

$$aRb \land bRc \Rightarrow aRc \ \forall a, b, c \in A.$$ 

Lemma 5. Consider a relation $R$ defined on set $A$. Suppose $\forall a \in A, \exists b \in A$ s.t. $aRb$. If $R$ is both symmetric and transitive, then $R$ is reflexive.

Proof. Consider $a, b \in A$ such that $(a, b) \in R$. If $b = a$, we are done. Otherwise, by symmetry $(b, a) \in R$. By transitivity, since $aRb$ and $bRa$, it must be that $aRa$. Since this holds for all $a \in A$, the relation is reflexive. □

Definition 5. A relation $R$ is irreflexive when no element $a$ relates to itself. Formally,

$$(a, a) \notin R \ \forall a \in A.$$ 

Note that a relation may be neither irreflexive nor reflexive.

Example 3: Suppose $A = \{1, 2\}$. Then $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Consider the following relations:

$$R_1 = \{(1, 2), (2, 1)\} \quad \text{irreflexive since } (1, 1), (2, 2) \notin R_1$$

$$R_2 = \emptyset \quad \text{irreflexive since } (1, 1), (2, 2) \notin R_2$$

$$R_3 = \{(1, 1), (1, 2), (2, 2), (2, 1)\} \quad \text{reflexive since } (1, 1), (2, 2) \in R_3$$

$$R_4 = \{(1, 1)\} \quad \text{neither reflexive nor irreflexive}$$

$R_4$ is not reflexive because $(2, 2) \notin R_4$. It is also not irreflexive because $(1, 1) \in R_4$. 

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Definition 6. A relation $R$ is asymmetric iff
\[(a, b) \in R \Rightarrow (b, a) \notin R \ \forall a, b \in A.\]

Notice that if a relation is asymmetric it cannot contain pairs of the form $(a, a)$ for any $a \in A$. Thus, if a relation is asymmetric then it is also irreflexive. Also notice that the only relation that is both symmetric and asymmetric is the empty set.

Example 4: Suppose $\mathbb{F}_2 = \{0, 1\}$. Then $\mathbb{F}_2 \times \mathbb{F}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$. Consider the following relations:

- $R_1 = \{(0,0)\}$  
  symmetric, transitive
- $R_2 = \emptyset$  
  symmetric, asymmetric, transitive
- $R_3 = \{((0,0), (0,1), (1,0), (1,1)\}$  
  symmetric, transitive
- $R_4 = \{(1,1), (0,0)(1,0)\}$  
  transitive

In $R_2$, there are no pairs $(a, b)$ in the relation for which we need to check the implications for symmetry, asymmetry, or transitivity. Recall that for a proposition $P \Rightarrow Q$, if $P$ is false, then the implication is true. Notice that $R_2$ is not reflexive, since $(0,0), (1,1) \notin R_2$. Relation $R_3$ is the entire Cartesian product. Thus, it is symmetric and transitive since every possible pair is present. Relation $R_4$ is not symmetric, as $(1,0) \in R_4$, but $(0,1) \notin R_4$. It is also not asymmetric, as it has pairs $(0,0)$ and $(1,1)$. To establish that $R_4$ is transitive, consider $(1,1)$ and $(1,0)$. Transitivity implies that $(1,0) \in R_4$, which is true. Also consider $(1,0)$ and $(0,0)$; we have that $(1,0) \in R_4$. Thus, for any pairs of the form $(a, b), (b, c) \in R_4$ we know that $(a, c) \in R_4$ as well.

Asymmetry is a strong condition; it does not allow any pairs of the form $(a, a)$ in the relation. A slightly weaker, but similar condition is anti-symmetry.

Definition 7. A relation $R$ is anti-symmetric iff
\[(a, b) \in R \text{ and } a \neq b \Rightarrow (b, a) \notin R \ \forall a, b \in A.\]

Any relation which is asymmetric is also anti-symmetric. For example, a relation defined by $<$ on the positive integers is both asymmetric and anti-symmetric. However, $\leq$ is not asymmetric, but it is anti-symmetric. We see this by observing that for any positive integer $x$, we have $x \leq x$ but $x \not< x$.

4 Summary

In this lecture, we used cardinality rules to prove theorems about relations defined on finite sets. We also considered the inverse relation. Finally, we studied relations defined on a set, and learned the rules for reflexivity, irreflexivity, symmetry, asymmetry, anti-symmetry, and transitivity.