Recall the following properties of relations $R$ from a set $A$ to a set $B$:

- **Function**: All elements of $A$ have out-degree $\leq 1$ (in the arrow diagram of $R$).
- **Total**: All elements of $A$ have out-degree $\leq 1$ (in the arrow diagram of $R$).
- **Injective**: All elements of $B$ have in-degree $\leq 1$ (in the arrow diagram of $R$).
- **Surjective**: All elements of $B$ have in-degree $\geq 1$ (in the arrow diagram of $R$).
- **Bijective**: All elements of $B$ have in-degree $= 1$ (in the arrow diagram of $R$).

Recall the properties of relations on sets (where $R$ is the relation on set $A$):

- **Reflexive**: $\forall a \in A. \ aRa$.
- **Irreflexive**: $\forall a \in A. \ \neg (aRa)$.
- **Transitive**: $\forall a, b, c \in A. \ aRb \land bRc \rightarrow a Rc$.
- **Symmetric**: $\forall a, b \in A. \ aRb \rightarrow b Ra$.
- **Asymmetric**: $\forall a, b \in A. \ aRb \rightarrow \neg (bRa)$.
- **Antisymmetric**: $\forall a, b \in A. \ aRb \land b Ra \rightarrow a = b$.

1. For any set $A$, we denote the power set of $A$ by $2^A$ and is defined to be the set of all subsets of $A$.

   (a) What are the elements of $2^A$ where $A = \{1, 2, 3\}$?
   (b) What are the elements of $2^A$ where $A = \{\emptyset, \{\emptyset\}\}$.
   (c) What is $|2^A|$ for any finite set $A$?

   **Solution:**

   (a) $2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
   (b) $2^A = \{\emptyset, \emptyset, \emptyset, \emptyset\}$.
   (c) $|2^A| = 2^{|A|}$.

2. For any sets $A, B, C,$ and $D$, what can be said about set $L = (A \cup B) \times (C \cup D)$ and $R = (A \times C) \cup (B \times D)$? Are they equal, is exactly one a subset of the other, etc.?
Solution: The answer is $R \subseteq L$ and $L \not\subseteq R$. To see the former, consider any element $(x, y) \in R$. By definition of $\cup$, $(x, y) \in A \times C$ or $(x, y) \in B \times D$. If $(x, y) \in A \times C$, then $x \in A$ and $y \in C$, so $x \in A \cup B$ and $y \in C \cup D$, and thus $(x, y) \in (A \cup B) \times (C \cup D)$. Similarly, if $(x, y) \in B \times D$, then $x \in B$ and $y \in D$, so $x \in A \cup B$ and $y \in C \cup D$, and thus $(x, y) \in (A \cup B) \times (C \cup D)$. In either case, $(x, y) \in (A \cup B) \times (C \cup D)$. Therefore, $R \subseteq L$.

To prove the latter, we can select the sets so that $L \not\subseteq R$; that is, we can ensure some element of $L$ is not in $R$. Consider $C = B = \emptyset$, and $A = \{a\}$ and $D = \{d\}$. Then $L = \{a\} \times \{d\} = \{(a, d)\}$, but $R = \emptyset \cup \emptyset = \emptyset$. Thus, for these choices of the sets, $L \not\subseteq R$.

3. Provide total functions $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$ with the following properties:

(a) $f$ is neither surjective nor injective.
(b) $f$ is surjective and not injective.
(c) $f$ is injective and not surjective.
(d) $f$ is surjective and injective.

Solution:

(a) $f(x) = 1$. Clearly $f$ is not surjective since there exists no $x \in \mathbb{Z}$ such that $f(x) = 2$ and $2 \in \mathbb{Z}^+$, and thus the in-degree of 2 is zero. Furthermore, $f$ is not injective since $f(1) = f(2) = 1$, and thus the in-degree of 1 is at least two.

(b) $f(x) = |x| + 1$. $f$ is not injective since $f(1) = f(-1) = 2$, so the in-degree of 2 is at least two. $f$ is surjective since for every $y \in \mathbb{Z}^+$, we can see that there is an $x \in \mathbb{Z}$ such that $f(x) = y$; in particular, this holds for $x = y - 1$.

(c) $f(x) = \begin{cases} 2x + 3 & \text{if } x \geq 0 \\ -2x & \text{otherwise} \end{cases}$

First, we prove that $f$ is not surjective. This consists of showing that the in-degree of some $y \in \mathbb{Z}^+$ is zero. In particular, this is true for (only) $y = 1$. To see this, suppose there was such an $x \in \mathbb{Z}$ such that $f(x) = 1$ for sake of contradiction. If $x \geq 0$, then $f(x) = 2x + 3 \geq 3$, which is a contradiction. Otherwise, if $x < 0$, then $x \leq 1$ and thus $f(x) = -2x \geq 2$ which is a contradiction. Thus, the in-degree of 1 is zero.

To prove that $f$ is injective, we show that the in-degree of every $y \in \mathbb{Z}^+$ is at most one. We’ve shown above that the in-degree of 1 is zero, so we only have to show this for the rest of the positive integers – those greater than two. First see that $f(x)$ is odd for any non-negative integer $x$, and that $f(x)$ is even for any negative integer $x$. Now suppose, for sake of contradiction, that there is some positive integer $y$ that is greater than two which has in-degree at least two. Then there exists distinct $a, b \in \mathbb{Z}$ such that $f(a) = f(b) = y$. From the previous observation, if $y$ is odd, then $a, b \geq 0$, and otherwise if $y$ is even, then $a, b < 0$. In the former case, then $f(a) = f(b)$ implies $2a + 3 = 2b + 3$ which only holds when $a = b$, a contradiction. In the latter case, then $f(a) = f(b)$ implies $-2a = -2b$ which is only holds when $a = b$, a contradiction. Thus, the in-degree of all $y \in \mathbb{Z}$ is at most one, so $f$ is surjective.
(d) \( f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ -2x & \text{otherwise.} \end{cases} \)

A similar proof for the injectivity of the function in part c implies that \( f \) is injective. The details are omitted here.

To see that \( f \) is surjective, we show that the in-degree of every \( y \in \mathbb{Z}^+ \) is at least one. Consider an arbitrary \( y \in \mathbb{Z}^+ \). If \( y \) is odd, see that \( y - 1 \) is odd and non-negative, so \( (y - 1)/2 \) is non-negative and integral. Thus, \( f((y - 1)/2) = y \). If \( y \) is even, then \( -y/2 \) is negative and integral. Thus, \( f(-y/2) = y \). In either case, there is some \( x \in \mathbb{Z} \) such that \( f(x) = y \), so the in-degree of all \( y \in \mathbb{Z}^+ \) is at least one. We conclude that \( f \) is surjective.

\[ \blacksquare \]

4. For each of the following relations, determine whether the relations are 1) reflexive, 2) irreflexive, 3) transitive, 4) symmetric, 5) antisymmetric, and 6) asymmetric. For parts d and e, list the elements of the relation.

(a) \( \emptyset \) on any non-empty set \( A \).
(b) \( A \times A \) on any non-empty set \( A \).
(c) \( \{(a,a), (a,b), (b,b), (b,c), (c,c)\} \) on set \( \{a,b,c,d\} \).
(d) \( \prec \) on set \( \{1,2,3,4\} \).
(e) \( \leq \) on set \( \{1,2,3,4\} \).
(f) \( \subseteq \) on set \( 2^A \) for any non-empty set \( A \).

Solution:

(a) The relation is irreflexive, transitive, symmetric, antisymmetric, and asymmetric. For completion, we prove that each property holds or does not hold for this relation. Let \( A \) be a non-empty set, and let \( R = \emptyset \).

i. Let \( a \) be an arbitrary element of \( A \). Since \( R \) does not contain \( (a,a) \), \( R \) is not reflexive.

ii. Since \( R \) does not contain \( (a,a) \) for any \( a \in A \), \( R \) is irreflexive.

iii. Since \( R \) is empty, there does not exist any \( a, b, c \in A \) such that \( aRb \wedge bRc \), and thus \( aRb \wedge bRc \rightarrow aRc \) is (vacuously) true for all \( a, b, c \in A \). Thus, \( R \) is transitive.

iv. Since \( R \) is empty, there does not exist any \( a, b \in A \) such that \( aRb \), and thus \( aRb \rightarrow bRa \) is (vacuously) true for any elements \( a, b \in A \). Thus, \( R \) is symmetric.

v. Since \( R \) is empty, there does not exist any \( a, b \in A \) such that \( aRb \wedge bRa \), and thus \( aRb \wedge bRa \rightarrow a = b \) is (vacuously) true for any elements \( a, b \in A \). Thus, \( R \) is antisymmetric.

vi. Since \( R \) is empty, there does not exist any \( a, b \in A \) such that \( aRb \), and thus \( aRb \implies \neg(bRa) \) is (vacuously) true for any elements \( a, b \in A \). Thus, \( R \) is asymmetric.

(b) The relation is reflexive, transitive, and symmetric. For completion, we prove that each property holds or does not hold for this relation. Let \( A \) be a non-empty set, and let \( R = A \times A \).
i. Since $aRb$ for all $a, b \in A$, clearly we have $aRa$ for all $a \in A$, and thus $R$ is reflexive.

ii. Since $R$ is reflexive, $R$ is not irreflexive.

iii. Consider any $a, b, c \in A$. Then we have $aRc$ since $xRy$ for all $x, y \in A$, and thus $aRb \land bRc \rightarrow aRc$ (regardless of whether $aRb \land bRc$ is true, which it is for this $R$). Thus, $R$ is transitive.

iv. Consider any arbitrary elements $a, b \in A$. Clearly $aRb$ and $bRc$, so $aRb \rightarrow bRc$ for all $a, b \in A$ and thus $R$ is symmetric.

v. Since $R$ is symmetric and $R$ is non-empty, there is some $a, b \in A$ such that $aRb \land bRa$ and $a \neq b$. Thus, $R$ is not antisymmetric.

vi. Since $R$ is not antisymmetric, $R$ is not asymmetric.

(c) The relation is antisymmetric.

(d) The relation is $\{ (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) \}$. It is irreflexive, transitive, asymmetric, and antisymmetric.

(e) The relation is $\{ (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 1), (2, 2), (3, 3), (4, 4) \}$. It is reflexive, transitive, and antisymmetric.

(f) The relation is reflexive, transitive, and antisymmetric.