Recall that a (undirected) graph $G$ is defined as an ordered pair $(V, E)$ where $V$ is a finite non-empty set, and $E \subseteq V^{(2)}$ is a set of edges (two-element subsets of $V$). The elements of $V$ are the vertices of $G$, and the elements of $E$ are the edges of $G$. It is convention to refer to $|V|$ as $n$ and $|E|$ as $m$.

An undirected bipartite graph $G$ is defined as an ordered pair $(V, E)$ where $V$ can be partitioned into two sets $A$ and $B$ such that $E \subseteq \{\{u, v\} \mid u \in A, v \in B\}$.

1. Prove by induction that, for any undirected graph $G = (V, E)$, the number of edges in $G$ is twice the sum of degrees of all vertices in $G$, i.e. $\sum_{v \in V} d(v) = 2m$.

**Solution:** We prove this by induction on the number of edges in the graph. That is, we will prove that for every non-negative integer $m$, the claim holds for any graph with $m$ edges.

Let $m$ be an arbitrary non-negative integer. If $m = 0$, then every vertex in any graph $G$ with zero edges has zero degree, so the claim holds in this case.

Otherwise, $m > 0$. Assume that the claim holds for all undirected graphs with less than $m$ edges. Let $e = \{a, b\}$ be some edge in $G$, and let $G' = (V, E \setminus \{e\})$ be the graph obtained by removing $e$ from $G$. Then $G'$ has $m - 1 < m$ edges, so $\sum_{v \in V} d'(v) = 2(m - 1)$ by the induction hypothesis where $d'(v)$ denotes the degree of $v \in V$ in graph $G'$. Note that $d'(u') = d(u)$ for every vertex $v \in V \setminus e$; that is, any vertex that is not an endpoint of $e$ has the same degree in $G$ and $G'$. Similarly, see that $d(a) = d'(a) + 1$ and $d(b) = d'(b) + 1$ since $a$ and $b$, the endpoints of edge $e$, have one more incident edge in $G$ than in $G'$, namely $e$. It follows that $\sum_{v \in V} d'(v) = 2 + \sum_{v \in V} d'(v) = 2 + 2(m - 1) = 2m$ as desired.

2. Prove by induction that, for any undirected bipartite graph $G = (V, E)$ with bipartition $A$ and $B$, $\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m$.

**Solution:** We will prove this by induction on the number of edges in the graph. That is, we will prove that for every non-negative integer $m$, the claim holds for any graph with $m$ edges.

Let $m$ be an arbitrary non-negative integer. If $m = 0$, then the degree of all vertices is also 0, so the claim holds in this case.

Otherwise, $m > 0$. Assume the claim holds for all undirected bipartite graphs with less than $m$ edges. Consider any undirected bipartite graph $G = (V, E)$ with bipartition $A$ and $B$ and $m$ edges. Let $e = \{a, b\}$ be an arbitrary edge in $G$ for some $a \in A$ and $b \in B$, and let $G'$ be the undirected graph obtained by removing edge $e$. Clearly $G'$ is bipartite with bipartition $A$ and $B$ and has $m - 1$ edges, so $\sum_{v \in A} d'(v) = \sum_{v \in B} d'(v) = m - 1$ by the induction hypothesis where $d'(v)$ denotes the degree of vertex $v$ in $G'$. Note that $d(v) = d'(v)$ for any vertex $v \notin \{a, b\}$, $d(a) = d'(a) + 1$, and $d(b) = d'(b) + 1$ since the only difference between $G$ and $G'$ is the edge $\{a, b\}$. It follows that $\sum_{v \in A} d(v) = 1 + \sum_{v \in A} d'(v)$ and $\sum_{v \in B} d(v) = 1 + \sum_{v \in B} d'(v)$ which immediately implies $\sum_{v \in A} d(v) = \sum_{v \in B} d(v)$ by the induction hypothesis. Finally, every vertex $v \in V$ is in exactly one of $A$ or $B$, so from problem 1 we have $\sum_{v \in A} d(v) + \sum_{v \in B} d(v) = 2m$. It follows that $\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m$ as desired.
3. Prove by induction that, for any binary string \( s \) that begins with a 1 and ends with a 0, there is a 1 immediately before a 0 somewhere in \( s \).

**Solution:** We prove this by induction on the length of the string. That is, we will prove that for every integer \( n \geq 2 \), the claim holds for any string of length \( n \).

Let \( n \) be an arbitrary positive integer. If \( n = 2 \), there is only one string of length 2 that begins with a 1 and ends with a 0, namely \( s = 0.1 \). Clearly the claim holds for \( s \) in this case.

Otherwise \( n > 2 \). Assume the claim holds for all strings that begin with a 1, end with a 0, and have length \( k \) such that \( 2 \leq k < n \). Let \( s \) be an arbitrary string of length \( n \) that begins with 1 and ends with 0. Since \( n > 2 \), \( s = 1.a.t.0 \) where \( a \in \{0,1\} \) and \( t \) is a string of length \( n - 3 \). There are two cases for \( a \):

(a) If \( a = 0 \), then \( s = 1.0.t.0 \) and the first 1 in \( s \) is immediately before the first 0 in \( s \), so we are done.

(b) If \( a = 1 \), then \( s = 1.1.t.0 \). Let \( u = 1.t.0 \) which has length \( n - 1 \), begins with a 1, and ends with a 0. Since \( 2 \leq n - 1 < n \), the induction hypothesis implies \( u \) contains a 1 immediately before a 0, and thus \( s = 1.u \) contains a 1 immediately before a 0.

4. A rooted binary tree is **full** if every node has either zero or two children. Prove that any rooted full binary tree with \( i \) internal nodes (those with at least one child) has \( 2^i + 1 \) total nodes.

**Solution:** We prove this by induction on the number of internal nodes. That is, we will prove that for every integer \( i \geq 0 \), the claim holds for any rooted full binary tree with \( i \) internal nodes.

Let \( i \) be an arbitrary non-negative integer. If \( i = 0 \), then the only tree with no internal nodes is the tree with a single root node which has no children. \( 2^0 + 1 = 1 \) so the claim holds in this case.

Otherwise, \( i > 0 \). Assume that the claim holds for all rooted full binary trees with less than \( i \) internal nodes. Let \( T \) be an arbitrary rooted full binary tree with \( i \) internal nodes. Since \( i > 0 \), the root node of \( T \) has children, specifically two since \( T \) is full. Let the subtrees rooted at the two children be \( T_1 \) and \( T_2 \), and let \( j \) be the number of internal nodes in \( T_1 \). Any non-root node of \( T \) is internal in \( T \) if and only the node is internal in \( T_1 \) or \( T_2 \). This implies the number of internal nodes in \( T_2 \) is \((i - 1) - j \). Since \( T \) is full, \( T_1 \) and \( T_2 \) must be full. Since \( 0 \leq j < i \) and \( 0 \leq i - 1 - j < i \), the induction hypothesis implies \( T_1 \) has \( 2j + 1 \) nodes and \( T_2 \) has \( 2(i - 1 - j) + 1 = 2i - 2j - 1 \) nodes. It follows that \( T \) has \((2j + 1) + (2i - 2j - 1) + 1 = 2i + 1 \) nodes.