Recall that an *(undirected) graph* \( G \) is defined as an ordered pair \((V, E)\) where \( V \) is a finite non-empty set, and \( E \subseteq V^{(2)} \) is a set edges (two-element subsets of \( V \)). The elements of \( V \) are the vertices of \( G \), and the elements of \( E \) are the edges of \( G \). It is convention to refer to \( |V| \) as \( n \) and \( |E| \) as \( m \).

An undirected *bipartite* graph \( G \) is defined as an ordered pair \((V, E)\) where \( V \) can be partitioned into two sets \( A \) and \( B \) such that \( E \subseteq \{\{u, v\} \mid u \in A, v \in B\} \).

A *bipartite matching* is a bipartite graph \( G = (V, E) \) where \( d(v) \leq 1 \) for all \( v \in V \). A bipartite matching with bipartition \((A, B)\) is \( A\)-perfect if \( d(v) = 1 \) for all \( v \in A \), and it is perfect if it is \( A\)-perfect and \( B\)-perfect; that is, \( d(v) = 1 \) for all \( v \in V \).

**Hall’s theorem:** A bipartite graph \( G = (V, E) \) with bipartition \((A, B)\) has a perfect \( A\)-matching if and only if \( |S| \leq |N(S)| \) for all \( S \subseteq A \).

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1. The standard (French) deck of playing cards has 4 *suits* of 13 *ranks* for a total of 52 cards. Suppose the cards are placed arbitrarily into a grid of 4 rows of 13 cards. Prove that, no matter how the cards are placed, there is a way to select one card from each column so that a card of each rank is selected.

**Solution:** Consider any arbitrary placement of the cards into the 4 by 13 grid. We say there is a solution in this grid if there is a way to choose a card from each column so that we have a card of each rank. A useful observation is that any subset of \( k \) cards has at most \( k \) distinct ranks, so it must be the case that in any solution all cards have different ranks. Thus, finding a solution is equivalent to selecting a card from each of the 13 columns so that all cards have different ranks.

Now consider the bipartite graph \( G = (C \cup R, E) \) where the vertices in \( C \) correspond to columns, the vertices in \( R \) correspond to the ranks of cards, and an edge is between column \( c \in C \) and rank \( r \in R \) if and only if there is a card of rank \( r \) in column \( c \). We will first show that a solution exists if and only if \( G \) has a perfect matching, then prove that \( G \) always contains a perfect matching; these facts together imply the grid always contains a solution as desired.

For the first part, consider any solution and denote by \( d_i \) the card chosen in the \( i \)th column, let \( r_i \) be the rank of card \( d_i \). The first fact observed implies \( r_i \neq r_j \) for any \( i \neq j \); that is, the ranks of any two cards in the solution are different. Now consider the subset \( E' \) of \( E \) containing the edges from column \( i \) to rank \( r_i \). Clearly \((C \cup R, E')\) is a perfect matching; indeed, every column vertex has degree 1 since exactly one card is chosen from each column in a solution, and every rank has degree 1 since there must be exactly one card with each rank in any solution. Now consider any perfect matching \( M = (C \cup R, E') \) of \( G \). Then for each edge in \( M \) is between a column \( i \) and some rank \( r_i \), so select any card of rank \( r_i \) from each \( i \)th column. Note that this must be possible by the definition of \( E \). Since each column and rank vertex has degree 1, exactly one card is chosen from each column, and they all have different ranks as desired.

For the second part, we want to show \( |S| \leq |N(S)| \) for all subsets of columns \( S \) to apply Hall’s theorem and conclude there is always a perfect matching in \( G \). Consider any subset \( S \) of \( k \) columns. There are only 4 cards of each rank, so there must be at least \( \lceil 4k/4 \rceil = k \) distinct ranks among the cards in the \( k \) columns. In other words, the maximum number of cards in a set of cards with \( d \) distinct ranks is at most \( 4r \), so if \( S \) had less than \( k \)
ranks among its cards, it must have at most \(4(k-1)<4k\) cards, a contradiction. \(|N(S)|\) is exactly the number of distinct ranks among the cards in the columns of \(S\), which must be at least \(k = |S|\). By Hall’s theorem, there is a perfect matching of columns to ranks, and since there are an equal number of ranks and columns, there is a perfect matching in \(G\).

2. Suppose you are given a connected undirected graph \(G = (V,E)\) and two coins \(c_1\) and \(c_2\) which are placed on two vertices \(a, b \in V\). A move consists of sliding both coins to a neighbor of the vertices they currently lie on. That is, if \(c_1\) and \(c_2\) are currently on vertices \(u_1\) and \(u_2\), we denote by \((u_1, u_2) \rightarrow (v_1, v_2)\) a move where \(\{u_1, v_1\}, \{u_2, v_2\} \in E\). The goal is to make a sequence of moves so that both coins lie on the same vertex; such a sequence is a solution. Prove that the only (connected) undirected graphs with no solutions are bipartite graphs where the coins start on vertices in different sets of the bipartition.

[Hint: First show there a solution if and only if there is an even-length walk between \(a\) and \(b\).]

**Solution:** First, we prove the hint. Suppose there is an even-length walk between \(a\) and \(b\), \(a = v_0, v_1, \ldots, v_{2k} = b\). Then the sequence of moves \((v_0, v_{2k}) \rightarrow (v_2, v_{2k-1}) \rightarrow \ldots \rightarrow (v_{k-1}, v_{k+1}) \rightarrow (v_k, v_k)\) is a solution.

Now suppose there is a solution \((a, b) = (u_1, u_2) \rightarrow (u_2, v_2) \rightarrow \ldots \rightarrow (u_{k-1}, v_{k-1}) = (u_k, v_k)\) where \(u_i, v_i \in V\) for all \(i \in \{1, 2, \ldots, k\}\) and \(u_k = v_k\). By the definition of move, \(\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\} \in E\) for all \(i \in \{1, 2, \ldots, k-1\}\). It follows that \(u_1, u_2, \ldots, u_{k-1}, v_k, v_{k-1}, \ldots, v_2, v_1\) is a walk between \(a\) and \(b\), and its length is twice the number of moves in the solution, so its length is even.

With the hint proven, consider the following two cases.

(a) Suppose \(G\) is not bipartite. Then \(G\) contains an odd-length cycle \(C\) (from lecture). Let \(y\) be an arbitrary vertex of \(Y\), \(P_a\) be any path between \(a\) and \(y\), and \(P_b\) any path between \(b\) and \(y\). If \(|P_a| + |P_b|\) is even, then the walk obtained by following \(P_a\) from \(a\) to \(y\) then following \(P_b\) from \(y\) to \(b\) is an even-length walk between \(a\) and \(b\) which implies a solution. Otherwise, if \(|P_a| + |P_b|\) is odd, let \(Z\) be the odd-length (closed) walk from \(c\) to \(c\) via the odd-length cycle \(Y\). Then the walk obtained by following \(P_a\) from \(a\) to \(y\), then \(Y\) from \(y\) to itself, then \(P_b\) from \(y\) to \(b\), is an even-length walk between \(a\) and \(b\) which implies a solution.

(b) Suppose \(G\) is bipartite. Any walk between two vertices on the same set of the bipartition has odd length, and any walk between two vertices in different sets of the bipartition has even length. Thus, there is a solution if and only if \(a\) and \(b\) lie in the same set of the partition. **Verify this fact by a short proof by induction.**

3. An undirected graph \(G = (V,E)\) is \(k\)-edge-connected if the subgraph \(G' = (V,E \setminus S)\) is connected for every \(S \subseteq E\) such that \(|S| < k\). Prove that any \(k\)-edge-connected graph has at least \(nk/2\) edges.
Solution: Let $G$ be a $k$-edge-connected graph. Every vertex of $G$ must have degree at least $k$, so the sum of vertex degrees is at least $nk$ which implies the number of edges is at least $nk/2$. ■